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Constructing decidable hybrid systems with velocity bounds

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Abstract— **In this paper, the question of bi-similarity between hybrid systems and their discrete quotients is studied from a new point of view. We consider two classes of hybrid systems: piecewise affine hybrid systems on simplices and piecewise multi-affine systems on multi-dimensional rectangles. Given a fixed partition of the state space, we derive sufficient conditions on the values of the vector fields at the vertices of the polytopes, in order that the constructed hybrid system is bi-similar with its corresponding discrete quotient transition system. The results are based on the fact that affine vector fields on simplices and multi-affine vector fields on rectangles are uniquely determined by their values at the vertices. In this way, an interesting class of decidable hybrid systems is determined. The result is applied to a motion planning problem for planar robots.**

I. INTRODUCTION

Systems that consist of a combination of continuous dynamics and discrete events are called *hybrid systems* [1], [2], [3], [4]. Continuous processes controlled by digital controllers are examples of such systems. In addition to discontinuities introduced by the computer, most physical processes exhibit discrete dynamics due to the action of elements ranging from valves, gears and switches in electromechanical systems to transcriptional regulators in genetic and metabolic networks. Hybrid systems are used as main modeling framework in a large number of areas such as automated highway systems, air-traffic management systems, embedded automotive and avionic controllers, manufacturing systems, robotics, genetic and metabolic networks, real-time communication networks, and real-time circuits. Formal verification is a very important issue during system design. The goal of formal verification is to prove that the system performs as expected. As the automated systems are growing in scale and complexity, the possibility of subtle errors becomes much larger. As a result, it is crucial to ensure that the system is always safe.

Formal analysis is concerned with *reachability analysis*, which is the problem of determining the set of states reached by a system starting from a given initial set, and *safety verification*, which is the problem of formally proving that a system does not have any trajectories connecting two given sets of states. A class of problems like the ones defined above is called *decidable*, if there exists a computational procedure that can decide, in a finite number of steps, whether any system in the class verifies any property in the class. For purely discrete systems described by finite state machines, decidability is an easy task, since it can be performed by exhaustively searching the state set. For hybrid and continuous systems, decidability is an important issue because the number of states in a continuous state set is uncountable.

In this paper, we consider a particular case of hybrid systems, that consist of specific dynamics (vector fields), defined in non-overlapping regions of the state space, called *invariants*. The decidability of such hybrid systems with given vector fields and given invariants is an important but difficult problem, that is not solved in this paper. Instead, we prove the decidability of a certain class of hybrid systems with prescribed invariants, but with arbitrary vector fields, restricted to a certain class. In other words, given the invariants, we want to construct vector fields so that the resulting hybrid system is decidable. This reverse engineering procedure is suggested by robotic motion planning problems, where a partition of the task space is naturally induced by the position and size of obstacles, and initial and goal regions, and vector fields have to be assigned to each of the regions so that the robots move from the initial to the final region while avoiding the obstacles and observing velocity bounds. The decidability of the corresponding hybrid systems reduces the motion planning problem to a search on a finite graph.

We focus on two classes of hybrid systems: triangular affine systems, *i.e.,* hybrid systems with triangular invariants and affine dynamics, and rectangular multi-affine systems, which are hybrid systems with rectangular invariants and multi-affine dynamics. There are several reasons for our choice of these classes of systems. First, given a polyhedral state set, triangulation and rectangular partition are the most attractive procedures for partitioning [5]. Second, affine vector fields are largely encountered in practice, as linearization of nonlinear systems around operating points (not necessarily equilibria). Third, nonlinear multi-affine dynamics are used for modeling in several application areas, ranging from biochemical networks [6], control of spacecraft and underwater vehicles [7], to competition and selection processes in economy and chemical networks. Moreover, affine systems on simplices and multi-affine systems on (multi-dimensional) rectangles, have some very interesting properties [8], [6] that can be used in the study of decidability problems for hybrid systems with these types of dynamics and invariants.

In this paper, we show that, if the triangular or rectangular invariants are given, the existence of affine or multi-affine dynamics rendering the corresponding hybrid systems decidable can be guaranteed by the nonemptiness of several polyhedral sets. We also provide formulas for the construction of the vector fields. These results are based on

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the fact that affine vector fields on simplices and multi-affine vector fields on multi-dimensional rectangles are uniquely determined by their values at the vertices. The values at all other points are convex combinations of the values at the vertices.

The paper is organized as follows. In Section II, we give definitions of hybrid systems, discrete quotients, and introduce the idea of simulation and bi-simulation. The problem is formulated in Section III. Affine hybrid systems with triangular invariants are treated in Section IV and multi-affine systems with rectangular invariants in Section V. An example of motion generation for a group of robots using decidable triangular affine systems is given in Section VI. The paper ends with conclusions and final remarks in Section VII.

II. BI-SIMILAR DISCRETE ABSTRACTIONS FOR HYBRID SYSTEMS

Formally, a hybrid system [9], [10] is defined as a tuple

$$HS = (\mathcal{X}, L, X_0, I, f, T), \tag{1}$$

where $\mathcal{X} \subseteq \mathbb{R}^N$, $N \in \mathbb{N}$ is the continuous state space, L is a finite set of locations (also called modes), $X = L \times \mathcal{X}$ is the overall state space of the system, $(l, x) \in L \times \mathcal{X}$ denotes its state, $X_0 \subseteq X$ is the set of initial states, I is the invariant, which assigns to each location $l \in L$ an invariant set $I(l) \subseteq \mathcal{X}$, $f : L \rightarrow (\mathcal{X} \rightarrow T\mathcal{X})$ is a mapping that specifies the continuous flow (vector field) in each location, and $T \subset L \times \mathcal{X} \times L$ is a set of discrete transitions. Motivated by robotic motion planing problems, we consider a special case of (1), where the invariants $I(l)$ are non-overlapping polyhedral regions in $\mathbb{R}^N$. In particular we assume that if the intersection $I(l_i) \cap I(l_j)$ of two polyhedral regions is nonempty, then it is a common face of $I(l_i)$ and $I(l_j)$. In this case, a transition T from l_i to l_j occurs when a state x flows through the boundary between $I(l_i)$ and $I(l_j)$.

The main idea in formal analysis is to be able to map the trajectories of a hybrid system to trajectories of a discrete system, *i.e.,* to transform a problem with an uncountable number of states to a decidable problem with finitely many states, that is equivalent to the first as far as reachability properties are concerned. This procedure is called *abstraction*.

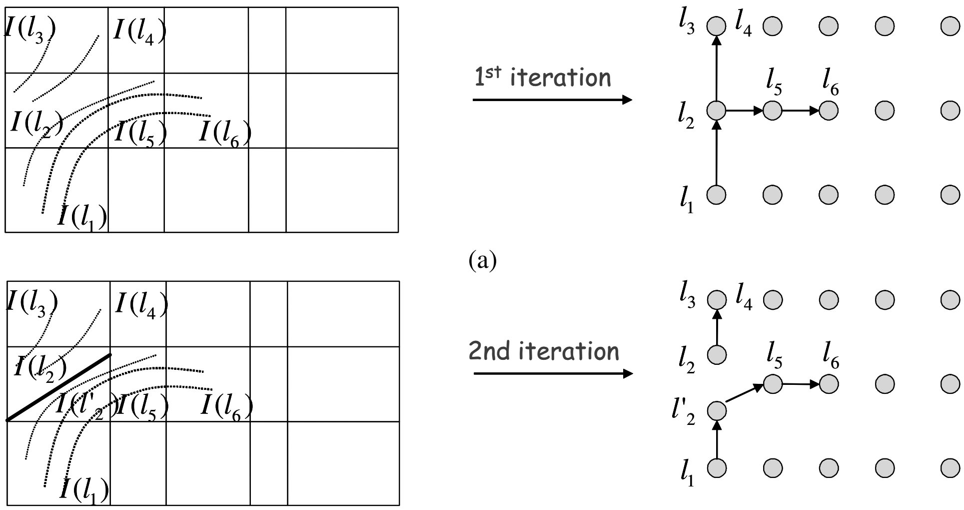
The *discrete quotient transition system* DS for the hybrid system HS defined in (1) is a tuple

$$DS = (L, L_0, t), \tag{2}$$

where L is the set of locations from the definition of HS, L_0 is the set of discrete initial states corresponding to X_0, and $t \subseteq L \times L$ is the set of transitions defined as follows: there exists a transition $t = (l, l')$ if and only if there exists $x \in \mathcal{X}$ so that (l, x, l') is a transition T of HS.

From this definition of DS, it is obvious to see that the discrete quotient system DS can reach everything that the initial hybrid system can reach, and can therefore be used for conservative reachability analysis, *i.e.,* to construct over-approximations of the reachable sets of HS. We say that DS *simulates* HS. However, the converse is in general not true. Indeed, it is easy to imagine that there are situations in which DS has trajectories that do not correspond to trajectories of HS. This can happen when different initial states in an arbitrary location $I(l)$ have different properties with respect to the reachability of the neighboring regions of $I(l)$. One such situation corresponds to the case when some initial states in $I(l)$ stay inside $I(l)$, while others leave $I(l)$, which makes HS and DS not equivalent with respect to reachability of neighbors. Another situation corresponds to the case when different initial states transit to different neighbors of $I(l)$. Even though HS and DS are equivalent with respect to reachability of neighbors of $I(l)$ (provided that no states stay inside $I(l)$ forever), the conservativeness appears while constructing the discrete quotient over several invariants. An illustration of this idea is given in Figure 1 (a), where the discrete trajectory $l_1 \rightarrow l_2 \rightarrow l_3$, which exists because of the definition of the discrete quotient, does not imply that there is a trajectory of HS passing through $I(l_1)$, $I(l_2)$, and $I(l_3)$. The degree of conservativeness increases with the dimension of the problem. This situation can be eliminated through refined partitioning, as shown in Figure 1 (b). If such an iterative refinement procedure terminates, *i.e.,* produces a discrete quotient with at most one transition from each discrete state, with the guarantee that all initial states in the corresponding invariant flow in finite time to the corresponding neighbor, HS and DS are called *bi-similar*, *i.e.,* they are equivalent with respect to reachability properties. The bi-simulation relation was first introduced in [11], [12], formally defined for linear control systems in [13], and for nonlinear systems in an abstract categorical context in [14].

In [15], it has been shown that reachability is undecidable for a very simple class of hybrid systems. Several decidable classes have been identified though by restricting the continuous behavior of the hybrid system, as in the case of timed automata [16], multirate automata [17], [18], and rectangular automata [15], [19], or by restricting the discrete behavior, as in order-minimal hybrid systems [20], [21], [22]. All these decidable classes are too weak to represent continuous and hybrid system models that arise in practice. Then one might be satisfied with sufficient abstractions, as the discrete quotient system defined by (2). But even finding the discrete quotient is not at all trivial. Related work focuses on partitioning using linear functions of the continuous variables, as in the method of predicate abstractions [23], [24], or using polynomial functions as in [24], [25]. However, to derive the transitions of the discrete quotient, one has to be able to either integrate the vector fields of the initial system [23], or use computationally expensive decision procedures such as quantifier elimination for real closed fields and theorem proving [24], which seriously limits the dimension of the problems that can be solved in one of these ways.



(a)

(b)

Fig. 1. The bi-simulation algorithm is an iterative refinement of partition, which terminates if, in the discrete quotient, there is at most one transition from each state: (a) DS simulates HS but HS does not simulate DS and (b) DS and HS are bi-similar.

III. PROBLEM FORMULATION

As stated in the Introduction, we will not address the decidability of hybrid systems with given vector fields and invariants in general, but rather characterize a class of decidable hybrid systems with given triangular or rectangular invariants and arbitrary affine or multi-affine vector fields. In other words, for these two classes of systems, given the invariants $I(l)$, $l \in L$, we want to construct vector fields f_l so that the resulting hybrid system HS (1) is bi-similar with its discrete quotient DS (2). Moreover, motivated by robotic motion planning problems, we impose polyhedral bounds for the vector fields:

Problem 1: Consider a polyhedral region $\mathcal{X}$ of $\mathbb{R}^N$ with a given triangulation or rectangular partition $I(l)$, $l \in L$. Let U be a polyhedral subset of $\mathbb{R}^N$. Characterize a class of hybrid systems HS with affine or multi-affine dynamics $f_l : I(l) \longrightarrow U$, that are guaranteed to be decidable without further refinement of the fixed partition $I(l)$, $l \in L$.

IV. TRIANGULAR AFFINE HYBRID SYSTEMS

Let $N \in \mathbb{N}$ and consider $N+1$ affinely independent points $v_1, \ldots, v_{N+1}$ in the Euclidean space $\mathbb{R}^N$, *i.e.,* there exists no hyperplane of $\mathbb{R}^N$ containing $v_1, \ldots, v_{N+1}$. Then the simplex S_N with vertices $v_1, \ldots, v_{N+1}$ is defined as the convex hull of $v_1, \ldots, v_{N+1}$:

$$S_N = \{x \in \mathbb{R}^N \,|\, x = \sum_{i=1}^{N+1} \lambda_i v_i,\ \sum_{i=1}^{N+1} \lambda_i = 1,\ \lambda_i \geq 0\} \tag{3}$$

For $i \in \{1, \ldots, N+1\}$, the convex hull of $\{v_1, \ldots, v_{N+1}\} \backslash \{v_i\}$ is a facet of S_N and is denoted by F_i. Let n_i denote the corresponding unit outer normal vector.

For $m \in \mathbb{N}$, let $f : \mathbb{R}^N \to \mathbb{R}^m$ be an arbitrary affine function

$$f(x) = Ax + b, \tag{4}$$

with $A \in \mathbb{R}^{m \times N}$ and $b \in \mathbb{R}^m$. Then we have:

Lemma 2: ([8, p. 26]) The affine function (4) is uniquely determined by its values $f(v_i) = g_i$, $i = 1, \ldots, N+1$ at the vertices of S_N. Moreover, the restriction of f to S_N is a convex combination of its values at the vertices and is given by:

$$f(x) = GV^{-1} \left[\begin{array}{c} x \\ 1 \end{array} \right],\ x \in S_N \tag{5}$$

where

$$G = [\ g_1 \ldots g_{N+1}\ ] \tag{6}$$

and

$$V = \left[\begin{array}{ccc} v_1 & \ldots & v_{N+1} \\ 1 & \ldots & 1 \end{array} \right] \tag{7}$$

are $m \times (N+1)$ and $(N+1) \times (N+1)$ real matrices.

Remark 3: The restriction of an affine function f to a facet F_i of S_N (i.e. F_i itself is a simplex in $\mathbb{R}^{N-1}$) is affine and for any $x \in F_i$, $f(x)$ is a convex combination of the values of f at the vertices of F_i.

Remark 4: Affine functions (4) defined on general full dimensional polytopes P_N are still convex combinations of their values at the vertices. However, the convex combinations are not unique and expression (5) for the construction of the affine function cannot be used, unless the polytope is triangulated, and (5) can be used in each simplex (see [8]).

In the rest of this section, we will restrict our attention to affine functions (4) with $m = N$ defined on a simplex S_N and with values in a polyhedral subset U of $\mathbb{R}^N$, *i.e.,* to affine vector fields with polyhedral bounds:

$$\dot{x} = f(x),\ f : S_N \to U \subseteq \mathbb{R}^N \tag{8}$$

Proposition 5: For any $i = 1, \ldots, N+1$, and any initial state in S_N, there is no trajectory of (8) leaving S_N through F_i if and only if $n_i^T f(v_j) \leq 0$, for all $j = 1, \ldots, N+1$, $j \neq i$.

Proof: For sufficiency, $n_i^T f(v_j) \le 0$, for all $j = 1,\dots,N+1$, $j \ne i$ implies $n_i^T f(x) \le 0$, for all $x \in F_i$, and therefore the system cannot cross facet F_i. The necessity is easily proved by contradiction. Suppose that there exists a vertex v_k, $k = 1,\dots,N+1$, $k \ne i$ so that $n_i^T f(v_k) > 0$, then, by continuity of f, there exists a whole neighborhood around v_k where $n_i^T f(v_k) > 0$. So there are initial states in this neighborhood such that the corresponding state trajectories leave S_N through F_i. ■

Proposition 5 can be used to provide a characterization of the requirement that an affine system can either stay inside a simplex forever, or drive all initial states in a simplex through a desired facet (*i.e.,* to a neighbor) in finite time. If one of these conditions is satisfied in all invariants of a triangular affine hybrid system HS , this hybrid system is bi-similar with its discrete quotient transition system DS, and therefore it is guaranteed that HS is decidable.

Specifically, the affine system (8), (4) starting in S_N will never leave S_N if and only if there exist $f(v_1),\dots,f(v_{N+1}) \in U$ so that for all $i = 1,\dots,N+1$ we have $n_i^T f(v_j) \le 0$, for all $j = 1,\dots,N+1$, $j \ne i$. These conditions can be equivalently formulated as feasibility checks at the vertices:

Proposition 6: There exists an affine vector field on S_N whose trajectories never leave S_N if and only if the following $N+1$ polyhedral sets are nonempty:

$$U_j = U \bigcap \{g \in \mathbb{R}^N | n_i^T g \le 0,\ i = 1,\dots,N+1,\ i \ne j\} \tag{9}$$

$j = 1,\dots,N+1$.

Also, it can be shown [8] that the affine vector field (8), (4) drives all initial states in the simplex S_N through a facet F_i, $i = 1,\dots,N+1$ in finite time if there exist $g_1,\dots,g_{N+1} \in U$ so that (1) $n_i^T g_j > 0$ for $j = 1,\dots,N+1$, and (2) $n_k^T g_j \le 0$ for all $k, j = 1,\dots,N+1$ with $k \ne i$, and $j \ne k$. As before, these conditions can be equivalently formulated at the vertices as follows:

Proposition 7: There exists an affine vector field (8) driving all initial states in the simplex S_N through the facet F_i in finite time if the following sets are nonempty:

$$\begin{aligned} U_i = U \bigcap \{g \in \mathbb{R}^N | n_j^T g \le 0, \quad & (10) \\ j = 1,\dots,N+1,\ j \ne i \text{ and } n_i^T g > 0\}, \quad & (11) \end{aligned}$$

$$\begin{aligned} U_j = U \bigcap \{g \in \mathbb{R}^N | n_i^T g > 0 \text{ and} \quad & (12) \\ n_k^T g \le 0 \text{ for all } k = 1,\dots,N+1,\ k \ne j,\ k \ne i\} \quad & (13) \end{aligned}$$

for all $j = 1,\dots,N+1$, $j \ne i$.

If the sets from Propositions 6 or 7 are all nonempty, then any choice of $g_i \in U_i$, $i = 1,\dots,N+1$ will give a valid affine vector field by formula (5). Indeed, for every $x \in S_N$, we know that $f(x)$ is a convex combination of $g_1,\dots,g_{N+1} \in U$. Hence, $f(x)$ is contained in the convex hull of $g_1,\dots,g_{N+1}$, which is the smallest convex set containing $g_1,\dots,g_{N+1}$, and therefore included in U. So the vector field is bounded everywhere in the simplex as required.

Propositions 6 and 7 provide a solution to Problem 1 for the case of triangular affine systems.

Theorem 8: Let $I(l)$, $l \in L$ be a given set of triangular invariants belonging to a hybrid system HS. Let $U \subset \mathbb{R}^N$ be a polyhedral set. If for every $l \in L$ there exists a vector field $f_l : I(l) \longrightarrow U$ satisfying either Proposition 6 or Proposition 7 with arbitrary exit facet F_i, and such that adjacent simplices do not have the same exit facet, then the corresponding hybrid system HS is bi-similar with its discrete quotient system DS, (and therefore decidable). Moreover, the bi-similarity of HS and DS can be shown without iterative refinement of the fixed partition $I(l)$, $l \in L$.

Note that in a worst case scenario, checking the sufficient conditions for bi-similarity between HS and its discrete quotient DS requires the application of Proposition 6 and Proposition 7 to each of the $N+1$ facets of each of the simplices $I(l)$, $l \in L$.

V. RECTANGULAR MULTI-AFFINE HYBRID SYSTEMS

An N-dimensional rectangle in $\mathbb{R}^N$ is characterized by two vectors $a = (a_1,\dots,a_N) \in \mathbb{R}^N$ and $b = (b_1,\dots,b_N) \in \mathbb{R}^N$, with the property that $a_i < b_i$ for all $i = 1,\dots,N$:

$$\begin{aligned} R_N \;=\; & \{x = (x_1,\dots,x_N) \in \mathbb{R}^N \mid a_i \le x_i \le b_i, \\ & i = 1,\dots,N\}. \end{aligned} \tag{14}$$

The set of 2^N vertices of R_N is denoted by V_N, and may be characterized as

$$V_N = \prod_{i=1}^{N} \{a_i, b_i\} \tag{15}$$

For $k = 1,\dots,N$, let $\xi_k : \{a_k, b_k\} \longrightarrow \{0,1\}$ denote the indicator function

$$\xi_k(a_k) = 0, \quad \xi_k(b_k) = 1, \quad k = 1,\dots,N. \tag{16}$$

Then R_N has $2N$ facets described by

$$F_N^{j,\xi_j(w_j)} = R_N \cap \{x \in \mathbb{R}^N \mid x_j = w_j\}, \tag{17}$$

with corresponding outer normals given by

$$n_N^{j,\xi_j(w_j)} = (-1)^{\xi_j(w_j)+1} e_j, \tag{18}$$

for all $w_j \in \{a_j, b_j\}$ and $j = 1,\dots,N$, where e_j, $j = 1,\dots,N$ denotes the Euclidean basis of $\mathbb{R}^N$.

A multi-affine function $f : \mathbb{R}^N \longrightarrow \mathbb{R}^m$ (with $N, m \in \mathbb{N}$) is a polynomial in the indeterminates $x_1,\dots,x_N$ with the property that the degree of f in any of the indeterminates $x_1,\dots,x_N$ is less than or equal to 1. Stated differently, f has the form

$$f(x_1,\dots,x_N) = \sum_{i_1,\dots,i_N \in \{0,1\}} c_{i_1,\dots,i_N} x_1^{i_1} \cdots x_N^{i_N}, \tag{19}$$

with $c_{i_1,\ldots,i_N} \in \mathbb{R}^m$ for all $i_1,\ldots,i_N \in \{0,1\}$ and using the convention that if $i_k = 0$, then $x_k^{i_k} = 1$.

Lemma 9: A multi-affine function (19) is uniquely determined by its values $f(v_1,\ldots,v_N)$ at the vertices of an N-dimensional rectangle R_N. Moreover, its restriction $f : R_N \longrightarrow \mathbb{R}^m$ is a (unique) convex combination of its values at the vertices:

$$f(x_1,\ldots,x_N) = \sum_{(v_1,\ldots,v_N)\in V_N} \prod_{k=1}^N \left(\frac{x_k-a_k}{b_k-a_k}\right)^{\xi_k(v_k)} \cdot \left(\frac{b_k-x_k}{b_k-a_k}\right)^{1-\xi_k(v_k)} f(v_1,\ldots,v_N) \tag{20}$$

The proof of the above Lemma can be found in [26].

Remark 10: The restriction of a multi-affine function f on R_N to a facet $F_N^{j,\xi_j(w_j)}$, $w_j \in \{a_j, b_j\}$, $j = 1,\ldots,N$ of R_N (which is a rectangle in $\mathbb{R}^{N-1}$) is itself a multi-affine function, and for each $x \in F_N^{j,\xi_j(w_j)}$, $f(x)$ is a convex combination of the values of f at the vertices of $F_N^{j,\xi_j(w_j)}$.

Using the property of Remark 10, the results of Propositions 5, 6, and 7 for affine systems on simplices, may be generalized to multi-affine systems with polyhedral bounds defined on rectangles:

$$\dot{x} = f(x),\ f : R_N \to U \subseteq \mathbb{R}^N \tag{21}$$

Proposition 11: For any $j = 1,\ldots,N$, and any $w_j \in \{a_j, b_j\}$, there is no trajectory of (21), (19) leaving R_N through $F_N^{j,\xi_j(w_j)}$ if and only if ${n_N^{j,\xi_j(w_j)}}^T f(v) \leq 0$, for all $v = (v_1,\ldots v_N) \in V_N$ with $v_j = w_j$.

Proposition 12: There exists a multi-affine vector field (21), (19) on R_N whose trajectories never leave R_N if and only if the following 2^N polyhedral sets are nonempty:

$$U_{(v_1,\ldots,v_N)} = U \cap \bigcap_{j=1}^N \{g \in \mathbb{R}^N | {n_N^{j,\xi_j(v_j)}}^T g \leq 0\} \tag{22}$$

for all $(v_1,\ldots,v_N) \in V_N$.

Proposition 13: There exists a multi-affine vector field (21), (19) driving all initial states in the rectangle R_N through an arbitrary exit facet $F_N^{j,\xi_j(w_j)}$ in finite time if the following 2^N sets are nonempty:

$$U_{(v_1,\ldots,v_N)} = U \bigcap \{g \in \mathbb{R}^N | {n_N^{j,\xi_j(w_j)}}^T g > 0 \text{ and } {n_N^{i,\xi_i(v_i)}}^T g \leq 0 \text{ for all } i = 1,\ldots,N,\ i \neq j\} \tag{23}$$

for all vertices $(v_1,\ldots,v_N) \in V_N$.

A proof of Proposition (13) can be found in [26].

Propositions 12 and 13 provide a solution to Problem 1 for rectangular multi-affine hybrid systems.

Theorem 14: Let $I(l), l \in L$ be a given set of rectangular invariants belonging to a multi-affine hybrid system HS. Let $U \subset \mathbb{R}^N$ be a polyhedral set. If for every $l \in L$ there exists a vector field $f_l : I(l) \longrightarrow U$ satisfying either Proposition 12 or Proposition 13 for an arbitrary exit facet F of the rectangle $I(l)$, and such that adjacent rectangles do not have the same exit facet, then the corresponding multi-affine hybrid system HS is bi-similar with its discrete quotient system DS, (and therefore decidable). Furthermore, the bi-similarity of HS and DS can be shown without iterative refinement of the fixed partition $I(l)$, $l \in L$.

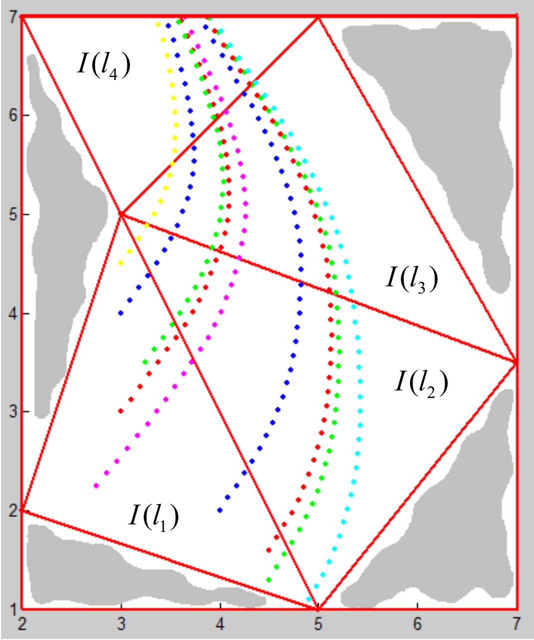


Fig. 2. A triangular partition of a planar environment. The shaded regions represent obstacles. Robots starting from arbitrary initial positions in the lower triangle are required to leave the rectangular region in finite time through the upper edge, while avoiding obstacles and observing velocity bounds. Nine sample trajectories are shown for illustration.

In a worst case scenario, checking the sufficient conditions of Theorem 14 for bi-similarity between HS and its discrete quotient DS requires the application of Proposition 12 and Proposition 13 to each of the $2N$ facets of each of the (multi-dimensional) rectangles $I(l)$, $l \in L$.

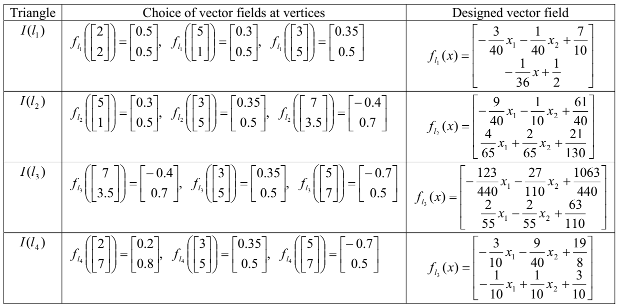
VI. MOTION PLANNING EXAMPLE

Consider a large number M of identical fully actuated planar robots described by control systems

$$\dot{x}^i = u^i,\ i = 1,\ldots,M,\ u^i \in U \tag{24}$$

where $x^i \in \mathbb{R}^2$ is the position vector of robot i in the world frame and $u^i \in U \subseteq \mathbb{R}^2$ is the corresponding control restricted to a rectangular set $U = [-1\,,\,1] \times [0\,,\,1]$, *i.e.,* the control magnitude on each axis is bounded to 1 and the robots are restricted to move in the direction of positive y.

The task is to generate feedback control laws $u^i(x^i)$ to move the robots from an initial to a final region of the task space in finite time, while avoiding obstacles and observing the velocity bounds $u^i \in U$. Assume that the initial region, the position and size of the obstacles, and the final region induce a triangular partition of the plane as shown in Figure 2. We solve this problem by constructing vector fields that obey the control restrictions everywhere in a triangle and drive all states in the initial triangle through the desired sequence corresponding to the task.



Triangle	Choice of vector fields at vertices	Designed vector field
$I(l_1)$	$f_{l_1}\left(\begin{bmatrix}2\\2\end{bmatrix}\right)=\begin{bmatrix}0.5\\0.5\end{bmatrix},\ f_{l_1}\left(\begin{bmatrix}5\\1\end{bmatrix}\right)=\begin{bmatrix}0.3\\0.5\end{bmatrix},\ f_{l_1}\left(\begin{bmatrix}3\\5\end{bmatrix}\right)=\begin{bmatrix}0.35\\0.5\end{bmatrix}$	$f_{l_1}(x)=\begin{bmatrix}-\frac{3}{40}x_1-\frac{1}{40}x_2+\frac{7}{10}\\-\frac{1}{36}x+\frac{1}{2}\end{bmatrix}$
$I(l_2)$	$f_{l_2}\left(\begin{bmatrix}5\\1\end{bmatrix}\right)=\begin{bmatrix}0.3\\0.5\end{bmatrix},\ f_{l_2}\left(\begin{bmatrix}3\\5\end{bmatrix}\right)=\begin{bmatrix}0.35\\0.5\end{bmatrix},\ f_{l_2}\left(\begin{bmatrix}7\\3.5\end{bmatrix}\right)=\begin{bmatrix}-0.4\\0.7\end{bmatrix}$	$f_{l_2}(x)=\begin{bmatrix}-\frac{9}{40}x_1-\frac{1}{10}x_2+\frac{61}{40}\\\frac{4}{65}x_1+\frac{2}{65}x_2+\frac{21}{130}\end{bmatrix}$
$I(l_3)$	$f_{l_3}\left(\begin{bmatrix}7\\3.5\end{bmatrix}\right)=\begin{bmatrix}-0.4\\0.7\end{bmatrix},\ f_{l_3}\left(\begin{bmatrix}3\\5\end{bmatrix}\right)=\begin{bmatrix}0.35\\0.5\end{bmatrix},\ f_{l_3}\left(\begin{bmatrix}5\\7\end{bmatrix}\right)=\begin{bmatrix}-0.7\\0.5\end{bmatrix}$	$f_{l_3}(x)=\begin{bmatrix}-\frac{123}{440}x_1-\frac{27}{110}x_2+\frac{1063}{440}\\\frac{2}{55}x_1-\frac{2}{55}x_2+\frac{63}{110}\end{bmatrix}$
$I(l_4)$	$f_{l_4}\left(\begin{bmatrix}2\\7\end{bmatrix}\right)=\begin{bmatrix}0.2\\0.8\end{bmatrix},\ f_{l_4}\left(\begin{bmatrix}3\\5\end{bmatrix}\right)=\begin{bmatrix}0.35\\0.5\end{bmatrix},\ f_{l_4}\left(\begin{bmatrix}5\\7\end{bmatrix}\right)=\begin{bmatrix}-0.7\\0.5\end{bmatrix}$	$f_{l_3}(x)=\begin{bmatrix}-\frac{3}{10}x_1-\frac{9}{40}x_2+\frac{19}{8}\\-\frac{1}{10}x_1+\frac{1}{10}x_2+\frac{3}{10}\end{bmatrix}$

Fig. 3. The choice of vector fields at the vertices of each triangle and the corresponding unique affine vector fields in each triangle.

Using Proposition 7 in each of the allowed triangular invariants $I(l_i)$, $i = 1, \ldots, 4$ (*i.e.,* triangles which are not occupied by obstacles), we derived necessary and sufficient conditions for the existence of affine vector fields (restricted to the polyhedral set U) driving all initial states through a separating facet in finite time. Our choice of vector fields at the vertices and the corresponding unique affine vector fields for each of the triangles are given in Figure 3.

Note that, for adjacent triangles, we chose the same velocity values at the vertices corresponding to the common facet. This guarantees the continuity of the vector field everywhere. Indeed, the vector fields in two adjacent triangles coincide on the separating facet, since their restrictions to the separating facet, which is a lower dimensional simplex, are uniquely determined by the values at the corresponding vertices. Therefore, the condition in Theorem 8 that adjacent simplices do not have the same exit facet is automatically satisfied.

The trajectories of $M = 9$ robots originating at arbitrary initial states in the initial triangle are shown for illustration in Figure 2.

VII. CONCLUSION

In this paper, we consider the problem of constructing vector fields with polyhedral bounds in each of the regions produced by a partition of a state space so that the resulting hybrid system is decidable. We consider two classes of hybrid systems, triangular affine systems and rectangular multi-affine systems, and show that the decidability of such systems is guaranteed if some specified polyhedral sets are nonempty. This reverse engineered approach to formal analysis of hybrid systems is illustrated in a robotic motion generation simulation example. Future work will be focused on property based reachability analysis, safety verification, and control of such systems, as well as on applications to motion and control problems in robotics.

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