

Controlling Three Dimensional Swarms of Robots

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Abstract— We address the problem of modeling and controlling a swarm of fully actuated point-like robots in three dimensions by generalizing the planar framework from [1]. We define a nine-dimensional abstraction of the swarm that has a product structure of the six-dimensional Euclidean group and a three-dimensional shape, and is independent of the number of robots. The group captures the pose of an ellipsoid spanning the swarm with semi-axes given by the shape variables. The overall abstract description is invariant to robot permutations. In addition, the shape is also invariant to left actions of the group. This description allows one to define and control the behavior of the swarm at a high level, with automatic generation of individual robot control laws. We present simulation results for controlling swarms of rotorcrafts.

I. INTRODUCTION

As a result of advances in computation, communication, sensor, and actuator technology, it is now possible to build teams of hundreds of small and inexpensive ground, air, and underwater robots. Such *swarms of autonomous agents* provide increased robustness to individual failures, the possibility to cover wide regions, and improved computational power through parallelism. However, planning and controlling such large teams of agents with limited communication and computation capabilities is a difficult problem that has received much attention from various communities. Even though, in some cases, it was observed or proved that local interaction rules in distributed natural or engineered multi-agent systems produce global behavior [6], [8], [16], some fundamental questions still remain to be answered. What is a good description that captures the essential features of a swarm? How do we specify its behavior? How can we reduce the size of the planning and control problem to make it computationally feasible?

One way of reducing the dimension of the control problem for large numbers of robots is to require them to conform to a rigid virtual structure [2], [5]. In this case, the motion planning problem is reduced to a left invariant control system on $SE(3)$ (or $SE(2)$ in the planar case), and the individual trajectories are $SE(3)$ ($SE(2)$) - orbits. Most of the recent works on stabilization and control of virtual structures model formations using *formation graphs* [3], [15], [17], [18]. The controllers guaranteeing local asymptotic stability of a given rigid formation can be derived using standard techniques such as input-output linearization [3], input-to-state stability [19], Lyapunov energy-type functions [4], [15], and biologically-

inspired artificial potential functions [14]. Virtual structures unnecessarily constrain the problem, making this approach inappropriate for tasks such as obstacle avoidance, passing of narrow tunnels, etc. Also, graph formulations and leader-follower architectures require identification and ordering of robots, which makes the overall architecture sensitive to failures.

The rigidity assumption is relaxed in an attempt to produce optimal trajectories in a geometrical framework in [2], however the amount of computation becomes prohibitively large. The notions of invariance with respect to world frames and permutations of robots are properly approached in shape theory in the well known “n-body problem,” where it is agreed that the coordinates divide into internal (shape) and orientational coordinates [7]. However, in application it is not clear how to construct shape coordinates explicitly, unless they are local or the problem is restricted to three or four bodies [10], [11], [13]. More recently, some of these ideas were applied to robotics [20]. However, this work is focused on constructing maximal shape spaces, and therefore is restricted to very small teams of robots. The problem of constructing an abstract description of a planar swarm with a product structure of a group and a reduced shape space whose dimension does not scale with the number of robots is approached in [1].

In this paper, we extend the results from [1] to three-dimensional spaces. For an arbitrarily large swarm of fully actuated robots in space, we construct a 9-dimensional abstraction A with a product structure of *group* G and *shape* S , i.e., $A = G \times S$. The group G is the 6-dimensional $SE(3)$, which captures the gross position and orientation of the swarm in a world frame, while the 3-dimensional shape space S is the general shape of the swarm. The overall abstract space is invariant to permutations of robots. Additionally, the shape is invariant to actions of $SE(3)$. We show that the swarm control problem can be correctly reduced to a left invariant control system in $SE(3)$ and the control of shape. Moreover, these features can be controlled independently. The abstraction that we construct provides a description of the region in the space occupied by the swarm in terms of a spanning ellipsoid with semi-axes proportional to s_1, s_2, s_3 and pose (R, d) in the world frame.

We will distinguish between high level planning or control via a supervisory agent and low level control for avoiding

collisions, maintaining formation and conforming to dynamic constraints. This distinction, which is also used by others (for example, [9]) is important for deriving decentralized control laws. Our work is unique in that our abstraction is derived from the two intrinsic geometric quantities, the Euclidean group variable, $g \in G$ and the shape, $s \in S$, and it provides a natural representation for planning at the higher level.

From a theoretical and computational point of view, generalizing the results from [1] to 3D raises several challenges. The main issue is that there is a fundamental difference between $SO(2)$, the one-dimensional group of rotations in the plane, and $SO(3)$, the 3-dimensional group of rotations in space. In contrast to [1], we are not able to derive an analytical expression for rotation in coordinates, and we do not provide a closed form expression for the abstraction map. However, we derive an expression for the tangent map connecting the velocity in the large dimensional space of the swarm and the control variables in the abstract space. From an application point of view, extending [1] to 3D gives one the opportunity to model and control spatial swarms, such as large groups of micro UAVs and UUVs.

The remainder of the paper is organized as follows: Section II details some notation and necessary preliminary definitions. Section III describes the general problem and defines a set of requirements to be resolved in the following sections. The abstraction is presented in Section IV. Sections V and VI provide the derivation of the control law and a discussion of the properties and effectiveness of the algorithm. Section VII presents simulation results. We conclude with final remarks in Section VIII.

II. NOTATION AND PRELIMINARIES

For any vector $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$, we denote by $\hat{\omega} \in \mathbb{R}^{3 \times 3}$ its skew-symmetric form, *i.e.*,

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \quad (1)$$

For $\zeta = (\omega, v) \in \mathbb{R}^6$, $\omega, v \in \mathbb{R}^3$, we use the notation $\hat{\zeta}$ to denote the following 4×4 matrix

$$\hat{\zeta} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}. \quad (2)$$

The set of all rotations in \mathbb{R}^3 is denoted by

$$SO(3) = \{R \mid R \in \mathbb{R}^{3 \times 3}, RR^T = I, \det R = 1\},$$

while the set of rigid body displacements in \mathbb{R}^3 is given by

$$SE(3) = \left\{ g \mid g = \begin{bmatrix} R & \mu \\ 0 & 1 \end{bmatrix}, R \in SO(3), \mu \in \mathbb{R}^3 \right\}. \quad (3)$$

It is well known that both $SO(3)$ and $SE(3)$ are matrix Lie groups. The Lie algebras of $SO(3)$ and $SE(3)$, denoted by $so(3)$ and $se(3)$ [12] are defined as

$$so(3) = \{\hat{\omega} \mid \omega \in \mathbb{R}^3\} \quad (4)$$

and

$$se(3) = \{\hat{\zeta} \mid \zeta = (\omega, v), \omega, v \in \mathbb{R}^3\}, \quad (5)$$

respectively.

Given a curve $R(t) \in SO(3)$ an element $\hat{\omega}(t)$ of the Lie algebra $so(3)$ can be associated to the tangent vector $\dot{R}(t)$ at an arbitrary point t by:

$$\hat{\omega}(t) = R^T(t)\dot{R}(t). \quad (6)$$

Similarly, for $g(t) = (R(t), \mu(t)) \in SE(3)$ an element $\hat{\zeta}(t)$ of the Lie algebra $se(3)$ can be associated to $\dot{g}(t)$ by:

$$\hat{\zeta}(t) = g^{-1}(t)\dot{g}(t) \quad (7)$$

Conversely, equations (6) and (7), written as $\dot{R} = R\hat{\omega}$ and $\dot{g} = g\hat{\zeta}$, can be seen as a left invariant control system on $SO(3)$ and $SE(3)$, respectively.

III. PROBLEM FORMULATION AND APPROACH

Consider a set of N point-like robots described by position vectors q_i , $i = 1, \dots, N$ in a world frame $\{W\}$ of a three-dimensional Euclidean space \mathbb{R}^3 . We assume that the robots appear at a high level as kinematically controlled and fully actuated, *i.e.*, they are described by control systems of the form

$$\dot{q}_i = u_i, \quad i = 1, \dots, N, \quad (8)$$

where $q_i \in \mathbb{R}^3$, $u_i \in \mathbb{R}^3$. We collect all the robot states in $q = [q_1^T, \dots, q_N^T]^T \in \mathbb{R}^{3N}$ (referred to as the configuration of the “swarm”) and the robot controls in $u = [u_1^T, \dots, u_N^T]^T \in \mathbb{R}^{3N}$. To recover the individual states and controls, we define the canonical projection $\pi_i(q) = q_i$, $\pi_i(u) = u_i$, $i = 1, \dots, N$. Equations (8) can therefore be written as:

$$\dot{q} = u. \quad (9)$$

Swarming tasks are usually specified in terms of a small set of essential features, while the exact behavior of each robot is not of interest. Our goal is to find an abstract description of the swarm that is rich enough to accommodate a large class of tasks, and small enough to make the planning and control problem computationally feasible. We look for abstractions which have a product structure of a Lie *group* capturing the pose of the team in the world frame and a *shape*, which is an intrinsic description of the team, invariant to the world coordinate frame. We will require that the group part can be controlled separately from shape. In our view, the idea of having a coordinate-free definition of the shape together with the fact that the shape and group control laws are decoupled is fundamental in formation control. Without knowing the coordinates of the robot in the world frame $\{W\}$, only the shape variables can be measured and controlled using on-board sensors of the robots.

Coordinates of geometrical objects describing the region occupied by the robots can serve as abstract descriptions of the team. For example, in [1], it is shown that a large class of planar tasks can be accomplished by controlling the pose (position of the center and orientation) and semi-axes

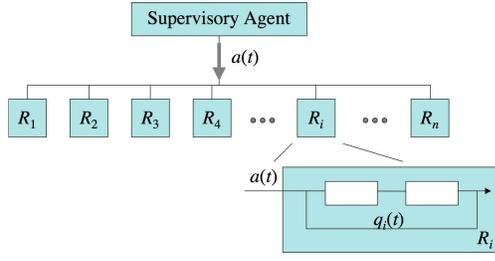


Fig. 1. Control and communication architecture. A supervisory agent prescribes the abstract state trajectory $a(t)$ and communicates that state to each robot R_i . Each robot requires knowledge of its position q_i and $a(t)$ in order to calculate and apply its individual control law.

of a spanning ellipse, while making sure that the robots stay inside it. In this case, the Lie group, $SE(2)$, is the set of all translations and rotations in plane and the shape is \mathbb{R}^2 . Inspired by this idea, in three dimensions, we look for *abstraction maps*

$$\phi : \mathbb{R}^{3N} \rightarrow SE(3) \times \mathbb{R}^3, \phi(q) = a = (g, s) \quad (10)$$

where a is called the *abstract state*, g is the *group variable*, and s is the *shape variable*. We require that g be the pose of a spanning ellipsoid, while s gives a description of its shape. The invariance property with respect to changes in the world frame can formally be written as

$$\phi(q) = (g, s) \Rightarrow \phi(\bar{g}q) = (\bar{g}g, s), \quad (11)$$

where $\bar{g}q$ represents the block diagonal action of the group element \bar{g} on the configuration $q \in Q$ and $\bar{g}g$ represents the left translation of g by \bar{g} using the composition rule on the group G .

To control the essential features (*i.e.*, the abstract state), we design feedback control laws for group $u_g : SE(3) \rightarrow \mathbb{R}^6$ and shape $u_s : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. The resulting control systems can now be written in the form

$$\dot{g} = g\hat{u}_g \quad (12)$$

and

$$\dot{s} = u_s, \quad (13)$$

respectively. The individual robot control laws (8) are sought in the form

$$u_i = u_i(q_i, a), \quad i = 1, \dots, N. \quad (14)$$

In other words, at each time instant the control for robot i depends on its own state q_i and the abstract state a , and does not explicitly depend on any of the other robot states. The dependence on a can be explicit or through the feedback control laws $u_g(q)$ and $u_s(s)$. While this approach appears to require a centralized communication architecture, it only requires partial state feedback. The abstract state can be estimated by a supervisory agent (for example, a blimp-like mother ship for a swarm of UAVs) as shown in Fig. 1. If the supervisor can measure and estimate (for example, using a camera and GPS) the abstract state, a , and broadcast it to all the robots, then each robot can compute its own control using

its state and equation (14). Alternatively, each robot can use its own sensors with a decentralized estimator to estimate the abstract state.

The two control systems (12) and (13) are called *decoupled* if, in the closed loop control system (8), (14), the group control u_g does not affect the shape s and the shape control u_s does not affect the group variables g through the map (10). If decoupling is achieved, one can move a swarm in space while preserving its shape, or can keep the swarm in place and change its shape.

Finally, since the abstract state captures the interesting features of the swarm, we want to make sure that, during the design process, we do not allow the swarm to spend energy in motions which are captured in the abstract space. In other words, we require that $\dot{a} = 0$ if and only if $\dot{q} = 0$. To summarize, in this paper we consider the following problem:

Problem 1: Find an abstraction (10) of the swarm and robot control laws (8) so that

- (i) The abstraction provides a description of the region in space spanned by the swarm and is invariant to robot permutations and changes in the world frame,
- (ii) The group and shape variables can be controlled separately,
- (iii) The velocity in the abstract space is zero if and only if the velocity of the swarm is zero,
- (iv) The control law of an individual robot depends only on its own state and the abstract state.

IV. ABSTRACTION AND PHYSICAL SIGNIFICANCE

We denote the group part $g \in SE(3)$ of the abstract state a as $g = (R, \mu)$, where $R \in SO(3)$ is the rotational part and $\mu \in \mathbb{R}^3$ is the translational part. For an arbitrary configuration $q \in Q$, we define the translational part of the group as

$$\mu = \frac{1}{N} \sum_{i=1}^N q_i \in \mathbb{R}^3. \quad (15)$$

Let

$$r_i = [x_i, y_i, z_i]^T = R^T(q_i - \mu), \quad i = 1, \dots, N. \quad (16)$$

The rotational part $R \in SO(3)$ (which is three-dimensional) is defined by the following three equations:

$$\sum_{i=1}^N x_i y_i = \sum_{i=1}^N x_i z_i = \sum_{i=1}^N y_i z_i = 0. \quad (17)$$

In this paper, we restrict our attention to a three-dimensional shape space $s = (s_1, s_2, s_3)$ defined by

$$s_1 = \sum_{i=1}^N \frac{x_i^2}{N-1}, \quad s_2 = \sum_{i=1}^N \frac{y_i^2}{N-1}, \quad s_3 = \sum_{i=1}^N \frac{z_i^2}{N-1}. \quad (18)$$

The dimension of the abstract space $SE(3) \times \mathbb{R}^3$ is therefore nine, independent of the number of robots. The rotation R defined by equations (17) can be seen as the rotation diagonalizing the inertia tensor of the system of particles q_i with respect to the centroid and orientation $\{W\}$, while $(N-1)s_1$,

$(N-1)s_2, (N-1)s_3$ are its eigenvalues. In this interpretation, the invariance of the abstraction with respect to changes in the world frame is apparent. Also, since only “democratic” sums over all robot indices are involved in equations (15), (17), and (18), it is easy to see that the definitions of group and shape are invariant to permutations of robots as well, as required in Problem (1).

Let

$$X_w = \sum_{i=1}^N (q_i - \mu)(q_i - \mu)^T \quad (19)$$

denote the matrix of second moments of the swarm in the world frame $\{W\}$. By (16) and (17), we have

$$X_w = RX_bR^T, \quad (20)$$

where

$$X_b = \sum_{i=1}^N r_i r_i^T = \sum_{i=1}^N \begin{bmatrix} x_i^2 & 0 & 0 \\ 0 & y_i^2 & 0 \\ 0 & 0 & z_i^2 \end{bmatrix}. \quad (21)$$

Although the abstraction does not require the swarm to be distributed in any particular manner, a normal distribution permits the definition of a physical interpretation that is useful in application and offers an intuitive understanding of the abstraction. Let

$$\Sigma = \frac{1}{N-1} X_w. \quad (22)$$

μ and Σ given by (15) and (22) can be interpreted as sample mean and covariance of a random variable with realizations q_i . If the random variable is known to be normally distributed, then, for a sufficiently large N , μ and Σ converge to the real parameters of the normal distribution. R defined by Eqns. (17) is the rotation that diagonalizes the covariance and s_1, s_2, s_3 are the eigenvalues of the covariance matrix. This means that, for a large number of normally distributed robots, $\mu, R, s_1, s_2,$ and s_3 give the pose and semiaxes of a concentration ellipsoid. Specifically, the contours of constant probability p for a trivariate normal distribution with mean μ and covariance Σ are described by

$$(x - \mu)^T \Sigma^{-1} (x - \mu) = -(3 \ln(2\pi) + 2 \ln(p\sqrt{s_1 s_2 s_3})). \quad (23)$$

The ellipsoid in (23), called the *equipotential* or *concentration ellipsoid*, has the property that p percent of the points are inside it and can thus be used as a spanning region for the robots, assuming that they are normally distributed. Therefore we can make the following statement: *p percent of a large number N of normally distributed robots described by a 9-dimensional abstract variable $a = (g, s) = (R, \mu, s_1, s_2, s_3)$ is enclosed in an ellipsoid centered at μ , rotated by $R \in SO(3)$ in the world frame $\{W\}$ and with semiaxes $\sqrt{cs_1}, \sqrt{cs_2}$, and $\sqrt{cs_3}$, where c is given by the right side of (23).*

V. CONTROL OF GROUP AND SHAPE

In this section, we construct the control systems for group (Eqn. (12)) and shape (Eqn. (13)), and discuss their decoupling as defined in Section III.

By differentiating (20) with respect to time, we get:

$$\dot{X}_w = \dot{R}X_bR^T + R\dot{X}_bR^T + RX_b\dot{R}^T. \quad (24)$$

The expansion of (24) using the equations (6), (18), and (21) yields,

$$R^T \dot{X}_w R = \dot{\omega} X_b + X_b \dot{\omega} + \dot{X}_b, \quad (25)$$

or

$$R^T \dot{X}_w R = (N-1) \begin{bmatrix} \dot{s}_1 & (s_1 - s_2)\omega_3 & (s_3 - s_1)\omega_2 \\ (s_2 - s_1)\omega_3 & \dot{s}_2 & (s_2 - s_3)\omega_1 \\ (s_1 - s_3)\omega_2 & (s_3 - s_2)\omega_1 & \dot{s}_3 \end{bmatrix}. \quad (26)$$

Note that each component of a is inherently an element of (26). Consider the following definition,

$$A_k^j = \frac{e_j^T R^T \dot{X}_w R e_k}{N-1}. \quad (27)$$

By varying j and k , the individual elements of the matrix on the right side of (26) are readily extracted. A few more definitions are helpful to reduce the complexity of the algebra necessary to show that the system is decoupled. Define

$$H_i^j = R e_i e_j^T R^T, T_j^i = H_j^i + H_i^j, \quad (28)$$

where $[e_1 e_2 e_3] = I_3$ and $i, j = \{1, 2, 3\}$. Note the following properties of these definitions (for $i \neq j$),

$$H_i^i = H_i = H_i^T, H_i H_j = 0, T_j^i = T_j^{iT} = T_i^j. \quad (29)$$

Additionally define $p_i = q_i - \mu$. Therefore, through algebraic simplification A_k^j is reduced to

$$\begin{aligned} A_k^j &= \frac{e_j^T R^T \dot{X}_w R e_k}{N-1} \\ &= \frac{1}{N-1} \sum_{i=1}^N [\dot{q}_i^T H_k^j p_i + p_i^T H_k^j \dot{q}_i] \\ &= \frac{1}{N-1} \sum_{i=1}^N p_i^T T_k^j \dot{q}_i. \end{aligned} \quad (30)$$

The result of (30) provides the time derivative of the abstract variables, less $\dot{\mu}$ which is

$$\dot{\mu} = \frac{1}{N} \sum_{i=1}^N \dot{q}_i. \quad (31)$$

The derivative of the shape variables are found to be, for $j = \{1, 2, 3\}$,

$$\dot{s}_j = \frac{2}{N-1} \sum_{i=1}^N p_i^T H_j \dot{q}_i, \quad (32)$$

and

$$\begin{aligned} \omega_1 &= \frac{\sum_{i=1}^N p_i^T T_3^2 \dot{q}_i}{(N-1)(s_2 - s_3)}, \omega_2 = \frac{\sum_{i=1}^N p_i^T T_3^1 \dot{q}_i}{(N-1)(s_3 - s_1)}, \\ \omega_3 &= \frac{\sum_{i=1}^N p_i^T T_2^1 \dot{q}_i}{(N-1)(s_1 - s_2)}. \end{aligned} \quad (33)$$

Equations (31), (32), and (33) allow the definition of a linear transformation between the time derivatives of the abstract variables and of the swarm configuration:

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \dot{\mu} \\ \dot{s}_1 \\ \dot{s}_2 \\ \dot{s}_3 \end{bmatrix} = \frac{1}{N-1} \begin{bmatrix} \frac{p_1^T T_3^2}{s_2 - s_3} & \cdots & \frac{p_N^T T_3^2}{s_2 - s_3} \\ \frac{p_1^T T_3^1}{s_3 - s_1} & \cdots & \frac{p_N^T T_3^1}{s_3 - s_1} \\ \frac{p_1^T T_2^1}{s_1 - s_2} & \cdots & \frac{p_N^T T_2^1}{s_1 - s_2} \\ \frac{N-1}{N} I_3 & \cdots & \frac{N-1}{N} I_3 \\ 2p_1^T H_1 & \cdots & 2p_N^T H_1 \\ 2p_1^T H_2 & \cdots & 2p_N^T H_2 \\ 2p_1^T H_3 & \cdots & 2p_N^T H_3 \end{bmatrix} \dot{q}. \quad (34)$$

Some simple calculations show that the linear map in (34) is full row rank unless $s_1, s_2, s_3 = 0$, which corresponds to the degenerate case when the swarm reduces to a point. If we exclude this situation, we can find the velocity of the swarm state as a function of the abstract controls by using the Moore Penrose inverse, corresponding to a minimum length solution in Euclidean metric. This results in the form:

$$\begin{aligned} \dot{q} &= \frac{s_2 - s_3}{s_2 + s_3} \omega_1 X_q^{\omega_1} + \frac{s_3 - s_1}{s_1 + s_3} \omega_2 X_q^{\omega_2} \\ &+ \frac{s_1 - s_2}{s_1 + s_2} \omega_3 X_q^{\omega_3} + \dot{\mu} X_q^\mu + \sum_{m=1}^3 \frac{\dot{s}_m}{2s_m} X_q^{s_m}, \end{aligned} \quad (35)$$

where (with $\{i, j, k\} = \{1, 2, 3\}, \{2, 1, 3\}, \{3, 1, 2\}$)

$$\begin{aligned} X_q^{\omega_i} &= [T_k^j(q_1 - \mu) \quad \cdots \quad T_k^j(q_N - \mu)]^T, \\ X_q^\mu &= [I_3 \quad \cdots \quad I_3]^T, \end{aligned} \quad (36)$$

and (with $j = \{1, 2, 3\}$)

$$X_q^{s_j} = [H_j(q_1 - \mu) \quad \cdots \quad H_j(q_N - \mu)]^T. \quad (37)$$

Each direction is independent of the other directions due to orthogonality, shown by the following equations:

$$\begin{aligned} H_i H_j &= 0 \text{ when } i \neq j, \\ H_i T_k^j &= 0 \text{ if } i \neq j, k, \\ H_i T_j^i &= H_j^i \text{ or } H_j T_j^i = H_j^i, \end{aligned} \quad (38)$$

$$\sum_{i=1}^N p_i^T H_j^i p_i = 0 \text{ when } i \neq j,$$

from (21) and (29). From (18) it is clear that each direction $X_q^{s_j}$ is orthogonal. Since $\sum_{i=1}^N p_i^T T_k^j T_n^m p_i$ is non-zero only if $j = n$ and $k = m$ the directions related to the group are also orthogonal. Additionally any combination of the group and shape directions, (i.e. $X_q^{\omega_j}$ or X_q^μ and $X_q^{s_j}$) is zero. Therefore the group and shape directions are also orthogonal from each other demonstrating that the abstraction is completely decoupled.

Using the canonical projection of (35), the individual control law for each agent as presented in (14) is found to be

$$\begin{aligned} u_i = \dot{q}_i &= \frac{s_2 - s_3}{s_2 + s_3} T_3^2 (q_i - \mu) \omega_1 + \frac{s_3 - s_1}{s_1 + s_3} T_3^1 (q_i - \mu) \omega_2 \\ &+ \frac{s_1 - s_2}{s_1 + s_2} T_2^1 (q_i - \mu) \omega_3 + \dot{\mu} + \sum_{k=1}^3 \frac{\dot{s}_k}{2s_k} H_k (q_i - \mu). \end{aligned} \quad (39)$$

Note that the controller for each agent is only dependent on its state q_i and the abstract state a which is requirement (iv) of Problem (1). A global observer that is able to acquire knowledge of the state and provide the values of the abstract state is sufficient to control the entire system.

From (34) and (35) it is clear that the abstraction variables are zero if and only if the configuration space is not changing as required by statement (iii) in Problem (1).

VI. CONTROL OF THE ABSTRACT STATE

Denote the desired abstract state trajectory by $a^{des}(t)$, the planned trajectory provided either by the supervisory agent or by a human commander. Because of the product structure, we can write:

$$a^{des}(t) = [g^{des}(t), s^{des}(t)].$$

We now design exponentially stabilizing controllers on $SE(3)$ and \mathbb{R}^3 for the group and shape variables. For (13),

$$u_s = \dot{s}^{des}(t) + K_s (s^{des}(t) - s(t)) \quad (40)$$

guarantees that the error in shape goes exponentially to zero provided K_s is chosen to be a 3×3 positive definite matrix. On $SE(3)$, we use the left-invariant kinematic metric for measuring errors between poses. Given the trajectory $g^{des}(t)$, the desired twist is given by

$$\hat{\xi}_g^{des}(t) = (g^{des}(t))^{-1} \dot{g}^{des}(t).$$

The error between poses g_1 and g_2 is given by the \log map on $SE(3)$: $e = \log(g_1^{-1} g_2)$ [12]. Thus, for (12), the closed-loop control is given by:

$$u_g = \xi^{des}(t) + K_g (e_g(t)) \quad (41)$$

where

$$\hat{e}_g(t) = \log(g^{-1}(t) g^{des}(t)).$$

This control law guarantees that the trajectory $g(t)$ converges exponentially to $g^{des}(t)$.

While we control the abstract description of the swarm to a desired behavior, as in input-output linearization problems, we need to make sure that the overall state of the swarm is bounded. In other words, we need to prove that bounds on the abstract state imply bounds on the state of the swarm. To this goal, assume that, for $k = \{1, 2, 3\}$,

$$\|\mu - \mu^{des}\| \leq M_\mu \quad (42)$$

$$|s_k - s_k^{des}| \leq M_{s_k}. \quad (43)$$

Note that

$$s_1 + s_2 + s_3 = \frac{1}{N-1} \sum_{i=1}^N (q_i - \mu)^T (q_i - \mu), \quad (44)$$

by the fact that $H_1 + H_2 + H_3 = I_3$. Considering equations (43) – (44) we see that

$$\begin{aligned} \|q_i - \mu\| &\leq \sqrt{(N-1)(s_1 + s_2 + s_3)} \\ &\leq \sqrt{(N-1)(M_{s_1} + M_{s_2} + M_{s_3} + s_1^{des} + s_2^{des} + s_3^{des})}, \end{aligned}$$

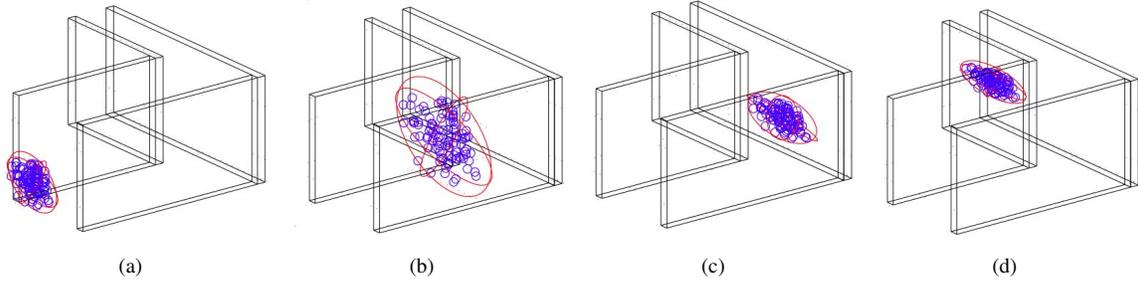


Fig. 2. The concentration ellipsoid surrounds the motion of 100 robots passing through a corridor. The trajectory for the system is defined in four segments. The first, 2(a), changes the position, orientation, and shape in the approach to the hallway. The second, 2(b), is a trajectory down the length of the hallway with vertical orientation. The third and final segments, 2(c)–2(d), are the reorientation, resizing, and passing of the swarm for the smaller hallway.

for $i = 1, \dots, N$. From (42),

$$\begin{aligned} \|q_i - \mu^{des}\| &= \|q_i - \mu + \mu - \mu^{des}\| \leq \|q_i - \mu\| + \|\mu - \mu^{des}\| \\ &\leq \sqrt{(N-1)(M_{s_1} + M_{s_2} + M_{s_3} + \bar{s}^{des})} + M_\mu, \end{aligned}$$

where $\bar{s}^{des} = s_1^{des} + s_2^{des} + s_3^{des}$. Therefore, assuming that the system is finite and the configuration of the swarm is initially bounded, the control algorithm will maintain the boundness of the system.

VII. SIMULATION RESULTS

In this section we present a simulation demonstrating the theoretical results discussed in this paper. Consider the task of driving a swarm of 100 robots, such as micro UAVs, through a constrained region similar to a series of hallways or corridors. In Fig. 2 we demonstrate the theoretic results of controlling the group through two hallways. The first hallway is considerably larger than the second hallway. The definition of the desired trajectories $a^{des}(t)$ is done in segments to emphasize the implication of the decoupled nature of the control law. In Fig. 2(a), the group is driven to an initial position, orientation, and shape in preparation for the first corridor. Note that the major axis of the concentration ellipse is vertically oriented while in Fig. 2(d) the major axis is oriented such that it is aligned with the direction of forward motion. After the initial formation the group variables are defined such that the swarm moves to the intersection of the hallways while the shape of the group is increased to leverage the available space (*i.e.* Fig. 2(b)). Fig. 2(c) depicts the transition as the swarm approaches the intersection, the orientation aligns with the walls of the hallway and the general size of the swarm is reduced. The final segment of the simulation maintains the orientation and shape of the swarm while varying $\mu^{des}(t)$ to traverse the length of the corridor (Fig. 2(d)).

VIII. CONCLUSION

In this paper we propose a method for controlling swarms of robots in three dimensions based on an abstraction that reduces the complexity of planning and control in the original high dimensional space to planning in a lower dimensional abstract space. The abstraction is invariant to the number and permutations of robots and permits decoupled control of the abstracted pose and shape of the large group of robots. The

individual control laws are dependent only on the robot's current position and the state of the abstraction.

ACKNOWLEDGMENT

This work was in part supported by: ARO Grants W911NF-05-1-0219, W911NF-04-1-0148.

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