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High Order Robust Adaptive Control Barrier Functions and Exponentially Stabilizing Adaptive Control Lyapunov Functions

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Abstract— **This paper studies the problem of utilizing data-driven adaptive control techniques to guarantee stability and safety of uncertain nonlinear systems with high relative degree. We first introduce the notion of a High Order Robust Adaptive Control Barrier Function (HO-RaCBF) as a means to compute control policies guaranteeing satisfaction of high relative degree safety constraints in the face of parametric model uncertainty. The developed approach guarantees safety by initially accounting for all possible parameter realizations but adaptively reduces uncertainty in the parameter estimates leveraging data recorded online. We then introduce the notion of an Exponentially Stabilizing Adaptive Control Lyapunov Function (ES-aCLF) that leverages the same data as the HO-RaCBF controller to guarantee exponential convergence of the system trajectory. The developed HO-RaCBF and ES-aCLF are unified in a quadratic programming framework, whose efficacy is showcased via two numerical examples that, to our knowledge, cannot be addressed by existing adaptive control barrier function techniques.**

I. Introduction

The problem of developing control policies that guarantee stability and safety of nonlinear control systems has received significant attention in recent years. In particular, the unification of Control Lyapunov Functions (CLFs) [1], [2] and Control Barrier Functions (CBFs) [3], [4] has provided a pathway towards safe and stable control of complex nonlinear systems such as autonomous vehicles [5], [6], multi-agent systems [7], and bipedal robots [8]. Although powerful, the guarantees afforded by these approaches are model-based, hence the success in transferring such guarantees to real-world systems is inherently tied to the fidelity of the underlying system model. Inevitably, such models are only an approximation of the true system due to parametric uncertainties and unmodeled dynamics, thus there is strong motivation to study the synthesis of CLF and CBF-based controllers in the presence of model uncertainty. Although robust approaches [9], [10] have demonstrated success in this regard, in general, such techniques can be highly conservative. On the other hand, data-driven approaches have demonstrated the ability to reduce uncertainty and yield high-performance controllers in terms of both safety and stability. Popular data-driven approaches for reducing uncertainty include work based on episodic learning [11], [12] or by modeling the uncertainty using Gaussian processes (GPs) [13], [14]. However, providing strong guarantees in an episodic learning setting is challenging and, although GP-based approaches account for very general classes of uncertainties, GPs can be computationally intensive and the generality offered by GPs generally results in probabilistic, rather than deterministic, guarantees on stability and safety.

As adaptive control [15] has a long history of success in controlling nonlinear systems with parametric uncertainty, there is also a rich line of work that unites CLFs and CBFs with techniques from adaptive control. The authors of [16] extend the adaptive CLF (aCLF) paradigm [15, Ch. 4.1] to CBFs, yielding the first instance of an *adaptive* CBF[1] (aCBF) that allows for the safe control of uncertain nonlinear systems with parametric uncertainty. The authors of [18]–[20] extend the aCBF techniques from [16] using set-membership identification, concurrent learning (CL) [21], [22], and hybrid techniques, respectively, which were shown to reduce the conservatism of original aCBF formulation. Nevertheless, all of the aforementioned aCBF techniques are limited to CBFs with relative degree one. In practice, however, many safety-critical constraints have relative degrees larger than one (e.g., constraints on the configuration of a mechanical system generally have at least relative degree two). The unification of CLFs/CBFs with techniques from CL adaptive control was also presented in [23], [24]; however, the resulting CLF controllers either only guarantee uniformly ultimately bounded stability or are limited to single-input feedback linearizable systems. Importantly, the CL-based aCBF controllers from [23], [24] do not provide strong safety guarantees since the CBF-based control inputs are generated using the estimated dynamics without accounting for estimation errors, leading to potential safety violations that can be understood through the notion of input-to-state-safety [25].

To address high relative degree safety constraints for systems with *known* dynamics, the authors of [26]–[29] introduce exponential and high order CBFs (HOCBFs), which provide a systematic framework to construct CBFs that account for high relative degree constraints. Importantly, as noted in [29], HOCBFs can also be used to simplify the search for valid CBFs since the dependence on higher order dynamics need not be directly encoded through the definition of the CBF itself (as in the relative degree one case). Rather, the dependence on higher order dynamics is *implicitly* encoded through conditions on higher order derivatives of the CBF candidate. Despite the advancements of both aCBFs and HOCBFs, to our knowledge, the intersection of these

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[1]The term aCBF was also used in [17] to refer to a class of CBFs that account for time-varying control bounds.

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two techniques has yet to be explored in the literature.

In this paper we unite aCBFs and HOCBFs to develop control policies satisfying high relative degree safety constraints for nonlinear systems with parametric uncertainty. Similar to [19], our approach leverages the concurrent learning technique presented in [22], which identifies uncertain parameters of the nonlinear system online by exploiting sufficiently rich data collected along the system trajectory. A key insight enabling our high relative degree approach is that if the relative degrees of the CBF with respect to the control and uncertain parameters are the same (in a sense to be clarified later in this paper), then the sufficient conditions for safety can be encoded through affine constraints on the control input, allowing control synthesis to be performed in a computationally efficient quadratic programming framework. Furthermore, unlike existing aCBF formulations for relative degree one constraints [16], [18]–[20], we show that our High Order Robust Adaptive Control Barrier Functions (HO-RaCBFs) inherit the robustness properties of zeroing CBFs [30] in the sense that our developed aCBFs not only render the safe set forward invariant, but also *asymptotically stable* when solutions begin outside the safe set. We then introduce a novel class of aCLF, termed exponentially stabilizing aCLFs (ES-aCLFs), that extends the CL paradigm from [22] to a CLF setting by exploiting the same history stack used to reduce conservatism of the safety controller to endow a nominal CLF-based control policy with *exponential* stability guarantees. The efficacy of the combined HO-RaCBF/ES-aCLF controller is demonstrated through simulations of a robotic navigation task and an inverted pendulum, the results of which can be found in an extended version of this paper [31].

II. Mathematical Preliminaries

Consider a nonlinear control affine system of the form

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

with state $x \in \mathbb{R}^n$ and control $u \in \mathcal{U} \subseteq \mathbb{R}^m$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipschitz vector fields modeling the drift and control directions, respectively. Given a feedback law $u = k(x,t)$, locally Lipschitz in x and piecewise continuous in t, the closed-loop vector field $f_{\text{cl}}(x,t) := f(x) + g(x)k(x,t)$ is also locally Lipschitz in x and piecewise continuous in t, implying that (1) admits a unique solution $x : \mathcal{I} \to \mathbb{R}^n$ starting from $x(0) \in \mathbb{R}^n$ on some maximal interval of existence $\mathcal{I} \subset \mathbb{R}_{\geq 0}$. A closed set $\mathcal{C} \subset \mathbb{R}^n$ is said to be *forward invariant* for the closed-loop system $\dot{x} = f_{\text{cl}}(x,t)$ if $x(0) \in \mathcal{C} \implies x(t) \in \mathcal{C}$ for all $t \in \mathcal{I}$. In this paper (and in the related literature) forward invariance is used to formalize the abstract notion of safety. Hence, if a given "safe" set $\mathcal{C}$ is forward invariant for $\dot{x} = f_{\text{cl}}(x,t)$, then we say the closed-loop system is *safe* with respect to $\mathcal{C}$. It will be assumed throughout this paper that any safe set $\mathcal{C}$ can be expressed as the zero-superlevel set of a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ as

$$\mathcal{C} = \{x \in \mathbb{R}^n \,|\, h(x) \geq 0\}. \tag{2}$$

A popular tool for developing controllers that render (2) forward invariant for (1) is the concept of a CBF [3], [4], which places Lyapunov-like conditions on the derivative of h to guarantee safety. A limitation of traditional CBFs from [3], however, is that their effectiveness is conditioned upon the assumption that the function h has relative degree one. Yet, many relevant systems and safe sets fail to satisfy such a condition, which has motivated the introduction of exponential and higher order CBFs [26]–[29] to account for safety constraints with *high relative degree*. Before proceeding, we recall that the *Lie derivative* of a differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ along a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined as $L_f h(x) := \frac{\partial h}{\partial x} f(x)$. This notation allows us to denote higher order Lie derivatives along an additional vector field g as $L_g L_f^{i-1} h(x) = \frac{\partial (L_f^{i-1}h)}{\partial x} g(x)$ (see e.g. [32, Ch. 13.2]).

Definition 1 ([32]). A sufficiently smooth function $h : \mathbb{R}^n \to \mathbb{R}$ is said to have *relative degree* $r \in \mathbb{N}$ with respect to (1) on a set $\mathcal{R} \subset \mathbb{R}^n$ if 1) for all $1 \leq i \leq r-1$, $L_g L_f^{i-1} h(x) \equiv 0$; 2) $L_g L_f^{r-1} h(x) \neq 0$ for all $x \in \mathcal{R}$.

To account for high relative degree safety constraints, the authors of [27]–[29] introduce the notion of a HOCBF. Before stating the definition, we recall from [3] that a continuous function $\alpha : (-b, a) \to (-\infty, \infty)$, for some $a, b \in \mathbb{R}_{>0}$, is said to be an *extended class $\mathcal{K}$ function* if it is strictly increasing and $\alpha(0) = 0$.

Definition 2 ([27]–[29]). Consider system (1) and a set $\mathcal{C} \subset \mathbb{R}^n$ as in (2). Let $\{\mathcal{C}_i\}_{i=1}^r$ be a collection of sets of the form $\mathcal{C}_i := \{x \in \mathbb{R}^n \,|\, \psi_{i-1}(x) \geq 0\}$, where $\psi_0(x) := h(x)$ and

$$\begin{aligned} \psi_i(x) &:= \dot{\psi}_{i-1}(x) + \alpha_i(\psi_{i-1}(x)),\ i \in \{1,\ldots,r-1\}, \\ \psi_r(x,u) &:= \dot{\psi}_{r-1}(x,u) + \alpha_r(\psi_{r-1}(x)), \end{aligned} \tag{3}$$

where $\{\alpha_i\}_{i=1}^r$ is a collection of differentiable extended class $\mathcal{K}$ functions. Then, the function h is said to be a HOCBF of order r for (1) on an open set $\mathcal{D} \supset \cap_{i=1}^r \mathcal{C}_i$ if h has relative degree r on some nonempty $\mathcal{R} \subseteq \mathcal{D}$ and there exists a suitable choice of $\{\alpha_i\}_{i=1}^r$ such that for all $x \in \mathcal{D}$

$$\sup_{u\in\mathcal{U}} \{\underbrace{L_f\psi_{r-1}(x) + L_g\psi_{r-1}(x)u + \alpha_r(\psi_{r-1}(x))}_{\psi_r(x,u)}\} \geq 0.$$

If $\mathcal{U} = \mathbb{R}^m$, the above states that h is a HOCBF if $\|L_g\psi_{r-1}h(x)\| = 0 \implies L_f\psi_{r-1}h(x) \geq -\alpha_r(\psi_{r-1}(x))$, implying that h need not have *uniform* relative degree on $\mathcal{D}$ as illustrated in [29], provided the unforced dynamics satisfy the above condition at points where $\|L_g\psi_{r-1}h(x)\| = 0$. The following result provides higher order conditions for safety.

Theorem 1 ([29]). *Let h be a HOCBF for* (1) *on $\mathcal{D} \subset \mathbb{R}^n$ as in Def. 2. Then, any locally Lipschitz controller $u = k(x) \in K_{cbf}(x)$, where $K_{cbf}(x) := \{u \in \mathcal{U} \,|\, \psi_r(x,u) \geq 0\}$, renders $\cap_{i=1}^r \mathcal{C}_i^r$ forward invariant for the closed-loop system.*

III. High Order Robust Adaptive Control Barrier Functions

This section introduces the concept of a High Order Robust Adaptive Control Barrier Function (HO-RaCBF), which provides a tool to synthesize controllers that guarantee the satisfaction of high relative degree safety constraints for nonlinear systems with parametric uncertainty. To this end, we now turn our attention to systems of the form

$$\dot{x} = f(x) + Y(x)\theta + g(x)u, \tag{4}$$

where f and g are known locally Lipschitz vector fields as in (1), $Y : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ is a known locally Lipschitz regression matrix, and $\theta \in \mathbb{R}^p$ is a constant vector of uncertain parameters. We assume that $f(0) = 0$ and $Y(0) = 0$ so that 0 is an equilibrium point of the unforced system. Our main objective is to synthesize controllers for (4) that guarantee safety under the presumption that the function h defining the safe set $\mathcal{C}$ as in (2) has a high relative degree with respect to (4). To facilitate our approach, we make the following assumption on the structure of the uncertainty in (4).

Assumption 1. Consider a set $\mathcal{C}$ as in (2) and an open set $\mathcal{D}$ as in Def. 2. If h has relative degree r on $\mathcal{R} \subseteq \mathcal{D}$ with respect to (4) (i.e., if there exist some nonempty $\mathcal{R} \subseteq \mathcal{D}$ such that $L_gL_f^{i-1}h(x) \equiv 0$ for all $1 \leq i \leq r-1$ and $L_gL_f^{r-1}h(x) \neq 0$ for all $x \in \mathcal{R}$), then there exists $\mathcal{R}' \subseteq \mathcal{D}$ such that $L_YL_f^{i-1}h(x) \equiv 0$ for all $1 \leq i \leq r-1$ and $L_YL_f^{r-1}h(x) \neq 0$ for all $x \in \mathcal{R}'$.

Remark 1. The above assumption requires that the uncertainty in (4) does not appear before the control when taking higher order derivatives of h. Although this may seem restrictive, a variety of physical systems satisfy Assumption 1. Examples include Lagrangian mechanical systems, where h is a function of only the system's configuration. If the above assumption is not made, then the uncertain parameters θ will appear alongside the control input u in higher order terms, complicating the formulation of the affine constraints on u developed in this paper. From an adaptive control perspective, Assumption 1 is similar to the assumption that the uncertain parameters satisfy the matching condition.

According to Theorem 1, the forward invariance of $\mathcal{C}$ can be enforced by ensuring that the control input is selected such that the HOCBF condition from Def. 2 is satisfied for all $x \in \cap_{i=1}^{r}\mathcal{C}_i$; however, the presence of model uncertainty in (4) makes it impossible to directly enforce such a condition. To address this challenge, we aim to take a data-driven approach and update the estimates of the uncertain parameters online using techniques from adaptive control [15], [19], [21], [22] while guaranteeing safety at all times. Following the approach from [22], observe that integrating (4) over a time interval $[t-\Delta T, t] \subset \mathbb{R}$ using the Fundamental Theorem of Calculus allows (4) to be equivalently represented as

$$\Delta x(t) = x(t) - x(t-\Delta T) = \mathcal{F}(t) + \mathcal{Y}(t)\theta + \mathcal{G}(t), \tag{5}$$

where $\mathcal{F}(t) := \int_{t-\Delta T}^{t} f(x(\tau))d\tau$, $\mathcal{Y}(t) := \int_{t-\Delta T}^{t} Y(x(\tau))d\tau$, $\mathcal{G}(t) := \int_{t-\Delta T}^{t} g(x(\tau))u(\tau)d\tau$. Now let $\mathcal{H} := \{(t_j, x_j, x_j^-, u_j)\}_{j=1}^{M}$ be a history stack of $M \in \mathbb{N}$ instances of input-output data, where $t_j \in [\Delta T, t]$ denotes a sampling time, $x_j := x(t_j)$, $x_j^- := x(t_j - \Delta t)$, $u_j := u(t_j)$, and define[2]

$$\Lambda(t) := \sum_{j=1}^{M} \mathcal{Y}_j^\top \mathcal{Y}_j, \quad \lambda(t) := \lambda_{\min}(\Lambda(t)), \tag{6}$$

where $\lambda_{\min}(\Lambda)$ denotes the minimum eigenvalue of Λ. As noted in [19], [22], the function λ is piecewise constant between sampling times and nonnegative since $\Lambda(t)$ is at least positive semidefinite at all times. The following assumption will be used to ensure safety for all possible realizations of the uncertain parameters.

Assumption 2. The uncertain parameters θ belong to a known convex polytope $\Theta \subset \mathbb{R}^p$.

The above assumption implies that for any given parameter estimate $\hat{\theta} \in \Theta$ there exists some maximum possible estimation error $\tilde{\vartheta} \in \mathbb{R}^p$ in the sense that $\|\theta - \hat{\theta}\| \leq \|\tilde{\vartheta}\|$ for all $\hat{\theta} \in \Theta$. Given that Θ is a convex polytope, each component of $\tilde{\vartheta}$ can be computed as

$$\tilde{\vartheta}_i = \max\left\{\big|\min_{\theta,\hat{\theta}\in\Theta} \theta_i - \hat{\theta}_i\big|, \big|\max_{\theta,\hat{\theta}\in\Theta} \theta_i - \hat{\theta}_i\big|\right\},$$

where $|\cdot|$ denotes aboslute value and θ_i denotes the ith component of θ, which requires solving a pair of linear programs for each parameter. The following lemma, adapted from [19], provides a verifiable bound on the parameter estimation error.

Lemma 1 ([19]). *Consider system* (4) *and suppose the estimated parameters are updated according to*

$$\dot{\hat{\theta}} = \gamma \sum_{j=1}^{M} \mathcal{Y}_j^\top (\Delta x_j - \mathcal{F}_j - \mathcal{Y}_j\hat{\theta} - \mathcal{G}_j), \tag{7}$$

where $\Delta x_j := x_j - x_j^-$, $\mathcal{Y}_j := \mathcal{Y}(t_j)$, $\mathcal{F}_j := \mathcal{F}(t_j)$, $\mathcal{G}_j := \mathcal{G}(t_j)$, *and* $\gamma \in \mathbb{R}_{>0}$ *is an adaptation gain. Provided Assumption 2 holds and* $\hat{\theta}(0) \in \Theta$, *then the parameter estimation error* $\tilde{\theta} = \theta - \hat{\theta}$ *is bounded for all* $t \in \mathcal{I}$ *as*

$$\|\tilde{\theta}(t)\| \leq \nu(t) := \|\tilde{\vartheta}\| e^{-\gamma\int_0^t \lambda(\tau)d\tau}. \tag{8}$$

The above lemma implies that, under the update law in (7), the parameter estimation error is always bounded by a known value provided the initial parameter estimates are selected such that $\hat{\theta}(0) \in \Theta$. Moreover, if there exists some time T such that $\lambda(t) > 0$ for all $t > T$, then the bound in (8) implies all estimated parameters exponentially converge to their true values[3]. We now have the necessary tools in place to introduce a new class of aCBFs that allow for the consideration of high relative degree safety constraints.

Definition 3. Consider system (4) and a safe set $\mathcal{C} \subset \mathbb{R}^n$ as in (2). Consider a collection of sets $\{\mathcal{C}_i\}_{i=1}^{r}$ of the form $\mathcal{C}_i := \{x \in \mathbb{R}^n \,|\, \psi_{i-1}(x) \geq 0\}$, where $\psi_0(x) := h(x)$ and $\{\psi_i\}_{i=1}^{r}$

[2]The function Λ is implicitly a function of time as data is added/removed from the history stack $\mathcal{H}$ along the system trajectory.

[3]See [21, Ch. 3] for a discussion on the relation between traditional persistence of excitation conditions and the milder *finite* excitation conditions leveraged in concurrent learning adaptive control required to achieve $\lambda > 0$. We also refer the reader to [33] for various algorithms that record data points in $\mathcal{H}$ so as to ensure $\lambda(t)$ is always nondecreasing.

are defined as in (3). The sufficiently smooth function h is said to be a *high order robust adaptive control barrier function* (HO-RaCBF) of order r for (4) on an open set $\mathcal{D} \supset \cap_{i=1}^{r}\mathcal{C}_i$ if h has relative degree r on some nonempty $\mathcal{R} \subseteq \mathcal{D}$ and there exists a suitable choice of $\{\alpha_i\}_{i=1}^{r}$ as in (3) such that for all $x \in \mathcal{D}$, $\theta \in \Theta$, and $t \in \mathcal{I}$

$$\begin{aligned}\sup_{u\in\mathcal{U}} \{L_f\psi_{r-1}(x) + L_Y\psi_{r-1}(x)\theta + L_g\psi_{r-1}(x)u\} \\ \geq -\alpha_r(\psi_{r-1}(x)) + \|L_Y\psi_{r-1}(x)\|\nu(t),\end{aligned} \tag{9}$$

where ν is defined as in (8).

Intuitively, the above condition adds an additional buffer to the original HOCBF condition from Def. 2 to account for all possible realizations of the uncertain parameters given the set Θ. This buffer may shrink over time as the uncertain parameters are identified and exponentially converges to zero in the limit as $t \to \infty$ provided there exists a time T for which $\lambda(t) > 0$ for all $t \geq T$. Furthermore, Def. 3 allows us to consider the set of all control values satisfying (9) as

$$\begin{aligned}\hat{K}_{\text{cbf}}(x,\theta,t) \coloneqq \{u \in \mathcal{U} \mid & L_f\psi_{r-1}(x) + L_Y\psi_{r-1}(x)\theta \\ & + L_g\psi_{r-1}(x)u + \alpha_r(\psi_{r-1}(x)) \\ & - \|L_Y\psi_{r-1}(x)\|\nu(t) \geq 0\}.\end{aligned} \tag{10}$$

The following theorem shows that any well-posed control law $u = k(x,\hat{\theta},t)$ satisfying $k(x,\hat{\theta},t) \in \hat{K}_{\text{cbf}}(x,\hat{\theta},t)$ renders $\cap_{i=1}^{r}\mathcal{C}_i$ forward invariant for (4).

Theorem 2. *Consider system* (4)*, a set $\mathcal{C}$ defined by a sufficiently smooth function h as in* (2)*, and let h be a HO-RaCBF on $\mathcal{D}$. Provided Assumptions 1-2 hold and the estimated parameters are updated according to* (7)*, then any controller $u = k(x,\hat{\theta},t)$ locally Lipschitz in $(x,\hat{\theta})$ and piecewise continuous in t satisfying $k(x,\hat{\theta},t) \in \hat{K}_{cbf}(x,\hat{\theta},t)$ renders $\cap_{i=1}^{r}\mathcal{C}_i$ forward invariant for* (4).

Proof. According to Theorem 1, to guarantee forward invariance of $\cap_{i=1}^{r}\mathcal{C}_i$, it is sufficient to show that for each $x \in \cap_{i=1}^{r}\mathcal{C}_i$ the input is selected such as $\psi_r(x,u) \geq 0$. For the dynamics in (4), if Assumption 1 holds, then $\psi_r(x,u) = L_f\psi_{r-1}(x) + L_Y\psi_{r-1}(x)\theta + L_g\psi_{r-1}(x)u + \alpha_r(\psi_{r-1}(x))$. Hence, it is our aim to show that any $u = k(x,\hat{\theta},t)$ satisfying $k(x,\hat{\theta},t) \in \hat{K}_{\text{cbf}}(x,\hat{\theta},t)$ satisfies $\psi_r(x,k(x,\hat{\theta},t)) \geq 0$ for all $x \in \cap_{i=1}^{r}\mathcal{C}_i$, all $\hat{\theta} \in \Theta$, and all $t \in \mathcal{I}$. To this end, observe that under control $u = k(x,\hat{\theta},t) \in \hat{K}_{\text{cbf}}(x,\hat{\theta},t)$ (omitting functional arguments for ease of presentation)

$$\begin{aligned}\psi_r =& L_f\psi_{r-1} + L_Y\psi_{r-1}\hat{\theta} + L_Y\psi_{r-1}\tilde{\theta} \\ & + L_g\psi_{r-1}k + \alpha_r(\psi_{r-1}) \\ \geq& L_f\psi_{r-1} + L_Y\psi_{r-1}\hat{\theta} + L_g\psi_{r-1}k \\ & + \alpha_r(\psi_{r-1}) - \|L_Y\psi_{r-1}\|\|\tilde{\theta}\| \\ \geq& L_f\psi_{r-1} + L_Y\psi_{r-1}\hat{\theta} + L_g\psi_{r-1}k \\ & + \alpha_r(\psi_{r-1}) - \|L_Y\psi_{r-1}\|\nu \\ \geq& 0,\end{aligned}$$

for all $x \in \cap_{i=1}^{r}\mathcal{C}_i$, all $\hat{\theta} \in \Theta$, and all $t \in \mathcal{I}$. In the above, the first inequality follows from the fact that $L_Y\psi_{r-1}(x)\tilde{\theta} \geq -\|L_Y\psi_{r-1}(x)\|\|\tilde{\theta}\|$, the second from the bound in (8), and the third from (10). Since $\psi_r(x,k(x,\hat{\theta},t)) \geq 0$ holds for all $x \in \cap_{i=1}^{r}\mathcal{C}_i$, all $\hat{\theta} \in \Theta$, and all $t \in \mathcal{I}$, it follows from Theorem 1 that $\cap_{i=1}^{r}\mathcal{C}_i$ is forward invariant for the closed-loop system, as desired. $\square$

Definition 3 and Theorem 2 generalize the ideas introduced in [19] to constraints with high relative degree, thereby facilitating the application of such ideas to more complex systems and safe sets. Although the results of [19] apply to safe sets defined by multiple barrier functions, whereas ours apply only to those defined by a single barrier function, there exist various approaches in the CBF literature [34], [35] to formally combine multiple barrier functions[4] using smooth approximations of min/max operators [34] or nonsmooth analysis [35]. We now show that if the conditions of Theorem 2 hold on $\mathcal{D}$ and $x(0) \in \mathcal{D}\setminus \cap_{i=1}^{r}\mathcal{C}_i$, then any controller satisfying $k(x,\hat{\theta},t) \in \hat{K}_{\text{cbf}}(x,\hat{\theta},t)$ also guarantees asymptotic stability of $\mathcal{C}$, which ensures the developed controller is robust to perturbations [30].

Corollary 1. *Let the conditions of Theorem 2 hold and suppose that $x(0) \in \mathcal{D}\setminus \cap_{i=1}^{r}\mathcal{C}_i$. Provided $u = k(x,\hat{\theta},t) \in \hat{K}_{cbf}(x,\hat{\theta},t)$ renders the closed-loop dynamics* (4) *forward complete, then the set $\cap_{i=1}^{r}\mathcal{C}_i$ is asymptotically stable for the closed-loop system.*

A controller satisfying the conditions of Theorem 2 can be computed through the use of quadratic programming (QP). Specifically, given an estimate of the uncertain parameters $\hat{\theta}$ and a nominal feedback control policy $k_d(x,\hat{\theta},t)$, a minimally invasive safe controller can be computed using the following HO-RaCBF-QP

$$\begin{aligned}\min_{u\in\mathcal{U}} \quad & \tfrac{1}{2}\|u - k_d(x,\hat{\theta},t)\|^2 \\ \text{s.t.} \quad & L_f\psi_{r-1}(x) + L_Y\psi_{r-1}(x)\hat{\theta} + L_g\psi_{r-1}(x)u \\ & \geq -\alpha_r(\psi_{r-1}(x)) + \|L_Y\psi_{r-1}(x)\|\nu(t),\end{aligned} \tag{11}$$

which enforces the conditions of Theorem 2 provided the resulting controller is Lipschitz continuous and h is a valid HO-RaCBF with $u \in \mathcal{U}$. That is, the above QP (11) allows the nominal policy k_d to be executed on (4) if k_d can be formally verified as safe and intervenes in a minimally invasive fashion to guarantee safety only if k_d cannot be certified as safe. We illustrate in the following section how one can leverage the same history stack used to reduce uncertainty in the parameter estimates to synthesize a desired policy k_d with exponential stability guarantees.

IV. Exponentially Stabilizing Adaptive Control Lyapunov Functions

In this section we introduce the concept of an exponentially stabilizing adaptive control Lyapunov function (ES-aCLF) as a tool to exponentially stabilize uncertain nonlinear systems in the presence of parametric uncertainty.

[4]In practice, it is common to simply include multiple CBF-based constraints in a quadratic program such as the one proposed in (11).

Definition 4. A continuously differentiable positive definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be an *exponentially stabilizing adaptive control Lyapunov function* (ES-aCLF) for (4) if there exist positive constants $c_1, c_2, c_3 \in \mathbb{R}_{>0}$ such that for all $x \in \mathbb{R}^n$ and $\theta \in \mathbb{R}^p$

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2, \tag{12a}$$

$$\inf_{u\in\mathcal{U}}\{L_fV(x)+L_YV(x)\theta+L_gV(x)u\} \leq -c_3V(x). \tag{12b}$$

Given the above definition, let

$$\begin{aligned}K_{\text{clf}}(x,\theta) :=\{u \in \mathcal{U} \mid L_fV(x)\\ + L_YV(x)\theta + L_gV(x)u \leq -c_3V(x)\},\end{aligned} \tag{13}$$

denote the point-wise set of all control values satisfying (12b). The following lemma provides a parameter update law that can be combined with any locally Lipschitz control policy satisfying $k(x,\hat{\theta}) \in K_{\text{clf}}(x,\hat{\theta})$ to guarantee stability.

Lemma 2. *Consider system* (4)*. Let V be an ES-aCLF as in Def. 4 and define $z := \begin{bmatrix}x^\top & \tilde{\theta}^\top\end{bmatrix}^\top$. Provided the estimates of the unknown parameters are updated according to*

$$\dot{\hat{\theta}} = \Gamma L_YV(x)^\top + \gamma\Gamma\sum_{j=1}^{M}\mathcal{Y}_j^\top(\Delta x_j - \mathcal{F}_j - \mathcal{Y}_j\hat{\theta} - \mathcal{G}_j), \tag{14}$$

where $\Gamma \in \mathbb{R}^{p\times p}$ is a positive definite gain matrix and $\gamma \in \mathbb{R}_{>0}$ is a user-defined adaptation gain, then any locally Lipschitz controller $u = k(x,\hat{\theta})$ satisfying $k(x,\hat{\theta}) \in K_{clf}(x,\hat{\theta})$ ensures that the composite system trajectory $t \mapsto z(t)$ remains bounded in the sense that for all $t \in [0,\infty)$

$$\|z(t)\| \leq \sqrt{\tfrac{\eta_2}{\eta_1}}\|z(0)\|, \tag{15}$$

where $\eta_1 := \min\{c_1, \frac{1}{2}\lambda_{\min}(\Gamma^{-1})\}$ and $\eta_2 := \max\{c_2, \frac{1}{2}\lambda_{\max}(\Gamma^{-1})\}$ are positive constants. Moreover

$$\lim_{t\to\infty} x(t) = 0.$$

Proof. Consider the Lyapunov function candidate $V_a(z) := V(x) + \frac{1}{2}\tilde{\theta}^\top\Gamma^{-1}\tilde{\theta}$, which can be bounded for all $z \in \mathbb{R}^{n+p}$ as $\eta_1\|z\|^2 \leq V_a(z) \leq \eta_2\|z\|^2$. Taking the derivative of V_a along the composite system trajectory yields

$$\begin{aligned}\dot{V}_a =&L_fV(x) + L_YV(x)\theta + L_gV(x)u - \tilde{\theta}^\top L_YV(x)^\top\\ &- \gamma\tilde{\theta}^\top\sum_{j=1}^{M}\mathcal{Y}_j^\top(\Delta x_j - \mathcal{F}_j - \mathcal{Y}_j\hat{\theta} - \mathcal{G}_j)\\ =&L_fV(x) + L_YV(x)\hat{\theta} + L_gV(x)u - \gamma\tilde{\theta}^\top\Lambda(t)\tilde{\theta},\end{aligned} \tag{16}$$

where Λ is from (6). Using the fact that $\Lambda(t)$ is at least positive semi-definite for all time implies $\dot{V}_a$ can be bounded as $\dot{V}_a \leq L_fV(x) + L_YV(x)\hat{\theta} + L_gV(x)u$. Choosing $u = k(x,\hat{\theta}) \in K_{\text{clf}}(x,\hat{\theta})$ and the hypothesis that V is a valid ES-aCLF allows $\dot{V}_a$ to be further bounded as $\dot{V}_a \leq -c_3V(x) \leq 0$, revealing that $\dot{V}_a$ is negative semi-definite. Hence, V_a is nonincreasing and $V_a(z(t)) \leq V_a(z(0))$ for all $t \in [0,\infty)$, which can be combined with the bounds on V_a to yield (15). Since V is continuous and $\dot{V}_a \leq -c_3V(x) \leq 0$, it follows from the LaSalle-Yoshizawa theorem [15, Thm. A.8] that $\lim_{t\to\infty} c_3V(x) = 0$, implying $\lim_{t\to\infty} x(t) = 0$. □

The following theorem shows that if sufficiently rich data is collected along the system trajectory, then $x(t)$ and $\tilde{\theta}(t)$ both exponentially converge to the origin.

Theorem 3. *Under the assumption that the conditions of Lemma 2 hold, suppose that there exists a time $T \in \mathbb{R}_{\geq 0}$ and a positive constant $\underline{\lambda} \in \mathbb{R}_{>0}$ such that $\lambda(t) \geq \underline{\lambda}$ for all $t \in [T,\infty)$. Then, for all $t \in [0,T)$, $z(t)$ is bounded in the sense that* (15) *holds. Furthermore, for all $t \in [T,\infty)$, $z(t)$ exponentially converges to the origin at a rate proportional to $\eta_3 := \min\{\gamma\underline{\lambda}, c_1c_3\}$ in the sense that for all $t \in [T,\infty)$*

$$\|z(t)\| \leq \sqrt{\tfrac{\eta_2}{\eta_1}}\|z(T)\|e^{-\frac{\eta_3}{2\eta_2}(t-T)}. \tag{17}$$

Proof. Since $\lambda(t) \geq 0$ for all $t \in [0,T)$ the conclusions of Lemma 2 hold, implying $z(t)$ is bounded as in (15) for all $t \in [0,T)$. Provided $\lambda(t) \geq \underline{\lambda}$ for all $t \in [T,\infty)$ and $u = k(x,\hat{\theta}) \in K_{clf}(x,\hat{\theta})$ for all $(x,\hat{\theta}) \in \mathbb{R}^{n+p}$, then (16) can be bounded as

$$\dot{V}_a \leq -c_3V(x) - \gamma\underline{\lambda}\|\tilde{\theta}\|^2 \leq -\eta_3\|z\|^2 \leq -\frac{\eta_3}{\eta_2}V_a.$$

Invoking the comparison lemma [32, Lem. 3.4] implies $t \mapsto V_a(z(t))$ is bounded for all $t \in [T,\infty)$ as

$$V_a(z(t)) \leq V_a(z(T))e^{-\frac{\eta_3}{\eta_2}(t-T)},$$

which can be combined with the bounds on V_a to yield (17). □

Similar to the previous section, given an estimate of the uncertain parameters $\hat{\theta}$, control inputs satisfying (12b) can be computed using the following ES-aCLF-QP

$$\begin{aligned}\min_{u\in\mathcal{U}} \quad & \tfrac{1}{2}u^\top u\\ \text{s.t.} \quad & L_fV(x) + L_YV(x)\hat{\theta} + L_gV(x)u \leq -c_3V(x).\end{aligned} \tag{18}$$

A controller guaranteeing stability and safety[5] can then be synthesized by either taking the solution to (18) as k_d in (11) or by forming a single QP with the HO-RaCBF constraint from (11) and a relaxed version of the ES-aCLF constraint from (18) to guarantee feasibility provided the resulting controller is Lipschitz continuous.

Remark 2. Note that asymptotic stability of the origin for the closed-loop system (4) with $u = k(x,\hat{\theta}) \in K_{\text{clf}}(x,\hat{\theta})$ is guaranteed by Lemma 2 regardless of whether or not the richness of data condition $\lambda(t) \geq \underline{\lambda} > 0$ is satisfied. In this situation, the stability guarantees induced by the ES-aCLF reduce to those of the classically defined adaptive CLF from [15, Ch. 4.1] (see also [16]). In this regard, neither safety nor stability is predicated upon collecting sufficiently rich data — this data is exploited only to reduce conservatism of the

[5]Similar to works such as [16], [18], the parameter update laws for the proposed adaptive CBF and CLF are different — if one wishes to combine the two in a single QP-based controller, separate estimates of the uncertain parameters must be maintained. Despite this, note that the data from a single history stack can be used in both update laws.

HO-RaCBF controller and to endow the ES-aCLF controller with exponential convergence guarantees.

Remark 3. The concept of an ES-aCLF generalizes the adaptive control designs from [22] in that the Lyapunov functions used to verify stability of the controllers proposed in [22] meet the criteria of an ES-aCLF posed in Def. 4. We refer the reader to works such as [2] for a discussion on the potential advantages of using an optimization-based CLF-based control law as in (18) over a traditional closed-form feedback control law such as those posed in [22].

V. CONCLUSIONS

In this paper we introduced HO-RaCBFs and ES-aCLFs as a means to synthesize safe and stable control policies for uncertain nonlinear systems with high relative degree safety constraints. The novel class of HO-RaCBF is, to the best of our knowledge, the first to extend the aCBF paradigm from [16] to CBFs with arbitrary relative degree under mild assumptions regarding the structure of the uncertainty and, unlike existing formulations, the proposed HO-RaCBF inherits desirable robustness properties of zeroing CBFs [30]. The class of ES-aCLFs introduced herein builds upon the classical aCLF formulation [15, Ch. 4.1] by leveraging data-driven techniques from CL adaptive control to guarantee exponential stability. Directions for future research include relaxing Assumption 1 and extending the approach to systems with nonparametric and actuation uncertainty.

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