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On controlling aircraft and underwater vehicles

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Abstract— **In this paper we make the important observation that the attitude and angular velocity control systems for gas-jet aircraft and underwater vehicles are of a special form: they are affine control systems with constant control distributions and multi-affine drifts. For this general class of systems, we can construct (and check the existence of) bounded controls driving the system from initial to final regions of the state space. This can be used for tasks requiring repositioning or changing the velocity of the vehicle under constraints on both controls and state. We illustrate the procedure by solving the problem of changing the angular velocity of a parallelepiped aircraft under velocity and control constraints imposed by the task. The method should be seen as a "maneuver" procedure, allowing automatic generation of control laws for bringing the system in a desired region of its state space. If stabilization to a point is required, then locally stabilizing control laws can be used after the maneuver.**

I. Introduction

There has been a lot of research in the area of attitude and velocity stabilization and planning of rotating rigid bodies [1]. Some of the existing works assume that three torques are available for control, through gas jet actuators or momentum wheels. In this case the attitude stabilization problem can be solved by using linear [2] or nonlinear controllers [3]. If less than three control torques are available, then nonlinear control laws need to be employed [4], [5], [6]. Moreover, it can be proved that they are necessarily non-smooth [7]. The problem of reorientation of a rigid body can also be thought of as an interpolation problem in the special Euclidean group $SO(3)$ of rotations in three dimensions. Most of the works in this area extend ideas from interpolation in Euclidean spaces [8], [9] to curved spaces, by use of Bezier curves [10], [11], Bernstein polynomials [12], spatial rational B-splines [13], Hermite interpolation [14], etc. In contrast with these works, which are based on particular parameterizations of the group, coordinate free formulations and invariant solutions are proposed in [15], [16], [17]. Invariant variational approaches are used to generate optimal motions on $SO(3)$ in [18] and [19], [20], [21].

In this paper, we approach the rigid body control problem from a totally different perspective, using ideas and some of our results from formal analysis of continuous and hybrid systems [22], [23]. Our approach is somewhere in between stabilization to a point and interpolation between two end positions in the configuration space. We propose a non-smooth (but possibly continuous) feedback law which allows for a "maneuvering" procedure, *i.e.*, driving a rigid body attitude or angular velocity control system between arbitrary initial and final regions of the state space, while satisfying bounds on controls and state. An illustrative task that we can solve with this procedure is the following. Given an aircraft or underwater vehicle with gas jet actuators and physical bounds on the control torques, which is initially rotating at a certain angular velocity (not necessarily precisely known), we want to drive it towards a final, desired angular velocity. We also require that apriori given bounds on the velocity are satisfied during the transition. After the desired region of the state space is reached, we can use one of the locally stabilizing control laws in the works cited above, if convergence to a specific state is required. Of course we need to make sure that the local region of attraction includes the target region of our algorithm. Note that globally stabilizing controllers exist as well, but using those there is no way one can guarantee that the trajectories converging to a desired equilibrium satisfy certain properties. We believe that the satisfaction of bounds on controls and velocity makes the method very attractive in a large area of applications.

The main motivation for this paper is the important observation that a large class of mechanical control systems, including aircraft with gas-jet actuators and underwater vehicles, can be modelled as affine control systems $\dot{x} = f(x) + g(x)u$, where the drift $f(x)$ is multi-affine in the state x *i.e.*, affine in each component x_i of the state, and the control distribution $g(x)$ does not depend on x, *i.e.*, can be modelled as a constant matrix. Multi-affine functions have a very interesting property when restricted to hyper-rectangular regions of their domain [23]: they are uniquely determined by their values at the vertices and can be expressed as a convex combination of these values. This is the starting point towards our work on abstraction of multi-affine systems, which is the procedure of defining discrete transition systems with a finite number of states that can be used to decide their reachability and safety properties. In this paper, we provide a solution to a "maneuver" task by constructing a rectangular partition of the state space and solving the following control problem in each of the rectangles: drive all the states in the rectangle through an exit facet in finite time. The solution can be found by solving a set of linear inequalities in the controls at the vertices and it naturally arises as a multi-affine state feedback law satisfying the control bounds. The solution to the maneuver problem is a set of "stitched" rectangles connecting the initial to the final region and a non-smooth (but possibly continuous) feedback

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control law, which guarantees that all initial states in the initial region are driven to the final region in finite time, and all the possible trajectories are contained in the tube formed by the rectangles, therefore bounds on the states are easy to satisfy.

The rest of this paper is organized as follows. In Section II, we note that both gas jet actuated aircraft and underwater vehicles are modelled as multi-affine control systems. A comprehensive review of our results on controlling these kind of systems is given in Section III. A maneuver problem in the angular velocity space for a parallelepiped aircraft with gas jet actuators and bounds on both controls and velocity is solved in Section IV. The paper ends with concluding remarks.

II. Controlled aircraft and underwater vehicles

Consider an arbitrarily shaped aircraft with a body fixed frame $\{M\}$ in motion with respect to a world frame $\{W\}$. Let G be the inertia matrix of the aircraft with respect to its body frame and m its mass. Let $\zeta_1, \zeta_2, \ldots, \zeta_m$ be the axes about which the corresponding control torques $t_1, \ldots, t_m$ are applied by means of opposing pairs of gas jets. Let ω denote the angular velocity in body frame, v the translational velocity of the origin of the body in body coordinates, m the mass of the aircraft, and F the total force applied to the body at the center of mass expressed in body frame. Then, the kinematic equations of the aircraft can be written as

$$m\dot{v} = mv \times \omega + F \quad (1)$$

$$G\dot{\omega} = G\omega \times \omega + \sum_{i=1}^{m} \zeta_i t_i \quad (2)$$

Similarly, for an underwater vehicle modelled as a neutrally buoyant rigid body submerged in an ideal fluid, if the center of gravity of the vehicle coincides with the center of buoyancy, then the equations of motion can be written as [24]:

$$M\dot{v} = Mv \times \omega + F \quad (3)$$

$$G\dot{\omega} = G\omega \times \omega + Mv \times v + \sum_{i=1}^{m} \zeta_i t_i \quad (4)$$

where M is an added mass matrix which incorporates the mass of the body and the mass of the fluid replaced by the body [24] and all the remaining variables have the same significance as before.

The position and orientation in the world frame $\{W\}$ of both systems described above are identified with $SE(3)$, the Lie group of rigid body displacements in $\mathbb{R}^3$:

$$SE(3) = \left\{ A \,|\, A = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}, R \in SO(3),\, d \in \mathbb{R}^3 \right\}. \quad (5)$$

where d denotes the displacement of the origin of the body frame $\{M\}$ in $\{W\}$ and $R \in SO(3)$ its rotation:

$$SO(3) = \{R \,|\, RR^T = I,\, \det(R) = 1\} \quad (6)$$

The equations relating their positions and velocities are

$$\dot{R} = R\hat{\omega} \quad (7)$$

$$\dot{d} = Rv \quad (8)$$

where $\hat{()}$ is the skew symmetric operator.

To parameterize the rotation $R \in SO(3)$ we choose quaternions $q = (q_1, q_2, q_3, q_4) \in S^3$, where S^3 denotes the unit sphere in $\mathbb{R}^4$. Then, equation (7) can be written as:

$$\dot{q} = \frac{1}{2} Q(q) \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ 0 \end{bmatrix} \quad (9)$$

where

$$Q(q) = \begin{bmatrix} q_4 & -q_3 & q_2 & -q_1 \\ q_3 & q_4 & -q_1 & -q_2 \\ -q_2 & q_1 & q_4 & -q_3 \\ -q_1 & -q_2 & -q_3 & -q_4 \end{bmatrix} \quad (10)$$

and $(\omega_1, \omega_2, \omega_3)$ are the components of the angular velocity ω.

There are situations, especially in space missions, in which one is not interested in controlling the pose (displacement and rotation) of a spacecraft of underwater vehicle in a reference frame, but rather in regulating the body velocities of translation and rotation. In this case, equations (1) and (2), respectively (3) and (4), can be seen as control systems with states $x = (v, \omega)$ and controls $u = (F, t_1, \ldots, t_m)$. However, there are several situations in which one is interested in controlling only the attitude of a vehicle in a given world frame, and then equations (2) and (9) can be seen as a control system with state $x = (q, \omega)$ and control variables $u = (t_1, \ldots, t_m)$.

The motivation for this paper is the important observation that all the control systems mentioned above are affine control systems $\dot{x} = f(x) + g(x)u$ of a specific class: the drift vector field f is a sum of products of the state variables, and the control distribution $g(x)$ is constant, *i.e.*, $g(x)u$ is the span of a constant matrix. Therefore, let us consider the following class of control systems

$$\dot{x} = f(x) + Bu \quad (11)$$

The drift term $f : \mathbb{R}^N \to \mathbb{R}^N$ is a multi-affine function consisting of products of state variables (formally defined in (12)) and $B \in \mathbb{R}^{N \times m}$ is a constant matrix whose columns give the directly controllable directions. Since in real applications there are always physical bounds on the control variables, the input u is assumed to take values in a polyhedral set $U \subset \mathbb{R}^m$ only.

In the next section we review an important result we proved in [23]: to find a feedback control law $u(x) \in U$ driving all initial states in a hyperrectangle in $\mathbb{R}^N$ through a desired facet in finite time is equivalent to solving a set of linear inequalities in control variables at the vertices. Based on this result, one can design (and test the existence of) admissible control laws driving system (11) between given initial and final regions of the state space based on defining a rectangular partition of its state space and applying the above result several times. The initial region should be contained in the initial rectangle and the final region should contain the final rectangle. The satisfaction of trajectory constraints during the transition can be accommodated by choosing a certain sequence of adjacent rectangles connecting the initial to the final one.

III. Control of Multi-Affine Systems on Rectangles

Definition 1 (Multi-affine function): A multi-affine function $f : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a polynomial in the indeterminates $x_1, \ldots, x_N$ with the property that the degree of f in any of the indeterminates $x_1, \ldots, x_N$ is less than or equal to 1. Stated differently, f has the form

$$f(x_1, \ldots, x_N) = \sum_{i_1, \ldots, i_N \in \{0,1\}} c_{i_1, \ldots, i_N} x_1^{i_1} \cdots x_N^{i_N}, \quad (12)$$

with $c_{i_1, \ldots, i_N} \in \mathbb{R}^N$ for all $i_1, \ldots, i_N \in \{0, 1\}$ and using the convention that if $i_k = 0$, then $x_k^{i_k} = 1$.

An N-dimensional rectangle in $\mathbb{R}^N$ is characterized by two vectors $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$ and $b = (b_1, \ldots, b_N) \in \mathbb{R}^N$, with the property that $a_i < b_i$ for all $i \in \{1, \ldots, N\}$:

$$\begin{aligned} R_N(a,b) \;&=\; \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid \\ &\forall i \in \{1, \ldots, N\}: \; a_i \leq x_i \leq b_i\}. \end{aligned} \quad (13)$$

The set of vertices of $R_N(a,b)$ is denoted by $V_N(a,b)$, and may be characterized as

$$V_N(a,b) = \prod_{i=1}^{N} \{a_i, b_i\} \quad (14)$$

Let $\xi : \{a_1, \ldots, a_N, b_1, \ldots, b_N\} \longrightarrow \{0, 1\}$ be defined by

$$\xi(a_k) = 0, \quad \xi(b_k) = 1, \quad k = 1, \ldots, N \quad (15)$$

Then $R_N(a,b)$ has $2N$ facets described by

$$\begin{aligned} F_N^{j\xi(w_j)}(a,b) \;&=\; R_N(a,b) \cap \{x \in \mathbb{R}^N \mid x_j = w_j, \\ &w_j \in \{a_j, b_j\}, \; j = \overline{1,N} \end{aligned} \quad (16)$$

The outer normal of facet $F_N^{j\xi(w_j)}(a,b)$ is given by

$$n_N^{j\xi(w_j)} = (-1)^{\xi(w_j)+1} e_j, \; w_j \in \{a_j, b_j\}, \; j = \overline{1,N} \quad (17)$$

where e_j, $j = 1, \ldots, N$ denote the Euclidean basis of $\mathbb{R}^N$.

A rectangular partition of the state space $(x_1, \ldots, x_N)$ is defined by dividing Ox_i into $n_i \geq 1$ intervals by some points $0 = \theta_0^i < \theta_1^i < \ldots < \theta_{n_i}^i$. The j^{th} interval on the Ox_i, $i = \overline{1,N}$ axis is therefore defined as $\theta_{j-1}^i \leq x_i < \theta_j^i$, $j = \overline{1, n_i}$. By convention, $\theta_0^i = 0$ and $\theta_{n_i}^i$ is an upper bound giving a physical limit for x_i. The division of the axes determines a partition of the state space into $\prod_{i=1}^N n_i$ rectangles. If we let

$$\begin{aligned} a_{k_1 \ldots k_N} &= (\theta_{k_1-1}^1, \ldots, \theta_{k_N-1}^N) \in \mathbb{R}^N, \\ b_{k_1 \ldots k_N} &= (\theta_{k_1}^1, \ldots, \theta_{k_N}^N) \in \mathbb{R}^N \end{aligned} \quad (18)$$

then following the notation in (13), (14), and (16), an arbitrary rectangle from the partition is given by $R_N(a_{k_1 \ldots k_N}, b_{k_1, \ldots k_N})$, the corresponding set of vertices by $V_N(a_{k_1 \ldots k_N}, b_{k_1, \ldots k_N})$, and the facets by $F_N^{j\xi_j(w_j)}(a_{k_1 \ldots k_N}, b_{k_1, \ldots k_N})$. Explicitly,

$$\begin{aligned} R_N(a_{k_1 \ldots k_N}, b_{k_1, \ldots k_N}) = \{(x_1, \ldots, x_N) \in \mathbb{R}^N \mid \\ \theta_{k_i-1}^i \leq x_i \leq \theta_{k_i}^i, \; i = \overline{1,N}\} \end{aligned} \quad (19)$$

Remark 1: A convenient way of representing a rectangular partition of the state space (19) is as a *simple graph* with $\prod_{i=1}^N n_i$ nodes. Node $(k_1 \ldots k_N)$ corresponds to rectangle $R_N(a_{k_1 \ldots k_N}, b_{k_1, \ldots k_N})$. An edge in the graph connects nodes corresponding to adjacent rectangles, *i.e.*, there is an edge between any pair of nodes that differ by a Hamming distance of 1.

Problem 1 (Control): Consider a control system (11) with multi-affine drift (12) and a rectangular partition of its state space (19). Determine a feedback control law $u = u(x) \in U$ that drives all initial states from an initial rectangle through a given sequence of pairwise adjacent rectangles in finite time.

Using the graph formalism (Remark 1), Problem 1 requires that a given path in the graph be followed. To provide a solution to Problem 1, we first need to be able to design controls so that all initial states in a given rectangle are driven to an adjacent rectangle.

We can prove an interesting property [23] of multi-affine functions on rectangles, which is the basis for solving Problem 1: *a multi-affine function defined on a N-rectangle is uniquely determined by its values at the vertices. Moreover, inside the rectangle, the function is a convex combination of its values at the vertices.* Formally, we have:

Proposition 1: A multi-affine function $f : R_N(a,b) \longrightarrow \mathbb{R}^N$ is a convex combination of its values at the vertices of $R_N(a,b)$.

$$\begin{aligned} f(x_1, \ldots, x_N) = \textstyle\sum_{(v_1, \ldots, v_N) \in V_N(a,b)} \prod_{k=1}^N \left(\frac{x_k - a_k}{b_k - a_k}\right)^{\xi(v_k)} \\ \left(\frac{b_k - x_k}{b_k - a_k}\right)^{1-\xi(v_k)} f(v_1, \ldots, v_N), \end{aligned} \quad (20)$$

$$1 = \sum_{(v_1, \ldots, v_N) \in V_N(a,b)} \prod_{k=1}^N \left(\frac{x_k - a_k}{b_k - a_k}\right)^{\xi(v_k)} \left(\frac{b_k - x_k}{b_k - a_k}\right)^{1-\xi(v_k)}, \quad (21)$$

where $(v_1, \ldots, v_N) \in V_N(a,b)$.

The first step towards solving Problem 1 is designing a feedback controller so that all initial states in a given rectangle are driven through a desired facet in finite time. Let $R_N(a,b)$ be an N-dimensional rectangle in $\mathbb{R}^N$, and consider the control system (11) evolving in $R_N(a,b)$. The drift term $f : R_N(a,b) \rightarrow \mathbb{R}^N$ is a multi-affine function (12), $B \in \mathbb{R}^{N \times m}$ is a constant matrix whose columns give the directly controllable directions, and the input u is assumed to take values in a polyhedral set $U \subset \mathbb{R}^m$ only.

Problem 2 (Control to a facet): Consider the multi-affine system (11) on the rectangle $R_N(a,b)$, and let F be a facet of $R_N(a,b)$ with normal n pointing out of $R_N(a,b)$. For any initial state $x_0 \in R_N(a,b)$, find a time instant $T_0 \geq 0$ and an input function $u : [0, T_0] \longrightarrow U$, such that

(i) $\forall t \in [0, T_0] : \; x(t) \in R_N(a,b)$,

(ii) $x(T_0) \in F$, and T_0 is the smallest time-instant in the interval $[0, \infty)$ for which the state reaches the exit facet F,

(iii) $n^T \dot{x}(T_0) > 0$, i.e. the velocity vector $\dot{x}(T_0)$ at the point $x(T_0) \in F$ has a positive component in the direction of

n. This implies that in the point $x(T_0)$, the velocity vector $\dot{x}(T_0)$ points out of the rectangle $R_N(a,b)$.

Furthermore, this input function u should be realized by the application of a continuous feedback law

$$u(t) = k(x(t)), \tag{22}$$

with $k : R_N(a,b) \longrightarrow U$ a continuous function, that is independent of the initial state x_0.

Note that if the feedback law $k(x)$ in (22) is multi-affine, the closed-loop system is also multi-affine. The solution to Problem 2 is given by the following Theorem [23]:

Theorem 1: Let $R_N(a,b)$ be an N-dimensional rectangle in $\mathbb{R}^N$, and consider the multi-affine system

$$\dot{x} = f(x) + Bu, \qquad x(0) = x_0 \in R_N(a,b)$$

on $R_N(a,b)$, with $B \in \mathbb{R}^{N\times m}$, $f : R_N(a,b) \longrightarrow \mathbb{R}^N$ multi-affine, and $u \in U$, with $U \subset \mathbb{R}^m$ a polyhedral set. Let $F_N^{j\xi(w_j)}(a,b)$ be a facet of $R_N(a,b)$ for arbitrarily fixed $w_j \in \{a_j, b_j\}$ and $j \in \{1,\ldots,N\}$. If in every vertex $(v_1,\ldots,v_N) \in V_N(a,b)$ there exists an input $u_{(v_1,\ldots,v_N)} \in U$ such that

$$\begin{array}{ll} (1) & {n_N^{j\xi(w_j)}}^T (f(v_1,\ldots,v_N) + Bu_{(v_1,\ldots,v_N)}) > 0, \\ (2) & \forall k \in \{1,\ldots,N\}\backslash\{j\},\ \forall w_k \in \{a_k, b_k\} : \\ & {n_N^{k\xi(w_k)}}^T (f(v_1,\ldots,v_N) + Bu_{(v_1,\ldots,v_N)}) \leq 0, \end{array} \tag{23}$$

then there exists a multi-affine feedback solution to Problem 2 with exit facet $F_N^{j\xi(w_j)}(a,b)$ given by $u = k(x)$, where $k : R_N(a,b) \longrightarrow U$ is the multi-affine function uniquely determined by

$$\forall (v_1,\ldots,v_N) \in V_N(a,b) : \ k(v_1,\ldots,v_N) = u_{(v_1,\ldots,v_N)},$$

The function k can be constructed using formula (20).

Corrolary 1: If $>$ in (23) (1) is replaced by $\leq$, *i.e.,* the vector field is oriented towards the interior of the rectangle at all vertices, then the solution u guarantees that the closed loop system will never leave the rectangle $R_N(a,b)$.

Checking the sufficient conditions in formula (23) of Theorem 1 requires the solution of 2^N systems of $2N-1$ linear inequalities in m unknowns: for each vertex of $R_N(a,b)$, one system of $2N-1$ linear inequalities in the unknown $u \in \mathbb{R}^m$. If a solution exists, construction of a multi-affine feedback is immediate, using formula (20).

Remark 2: Conditions (1) and (2) in formula (23) of Theorem 1 provide polyhedral sets $U_{(v_1,\ldots,v_N)}$ of controls at the vertices $(v_1,\ldots,v_N) \in V_N(a,b)$ that solve Problem 2. If all the sets $U_{(v_1,\ldots,v_N)}$ are non-empty, then one can choose a representant $u_{(v_1,\ldots,v_N)}$ from each set and construct the feedback control using formula (20). Since each of $u_{(v_1,\ldots,v_N)} \in U$ and $u(x)$ is a convex combination of $u_{(v_1,\ldots,v_N)}$, then it is guaranteed that $u(x) \in U$ for all $x \in R_N(a,b)$. An interesting special case is when $\bigcap_{(v_1,\ldots,v_N)\in V_N(a,b)} U_{(v_1,\ldots,v_N)} \neq \emptyset$. An element $\bar{u} \in \bigcap_{(v_1,\ldots,v_N)\in V_N(a,b)} U_{(v_1,\ldots,v_N)}$ can be used as a constant (independent of the current state) control that solves Problem 2. Note that this is consistent with (20). Indeed, if $u_{(v_1,\ldots,v_N)} = \bar{u}$ for all $(v_1,\ldots,v_N) \in V_N(a,b)$, then $u(x) = \bar{u}$ due to (21). This case might be extremely useful for practical situations when the state is not available for feedback.

Remark 3: A solution to Problem 1 involves a choice of a sequence of adjacent rectangles (nodes in the corresponding graph) and the application of Theorem 1 (which is a solution to Problem 2) to drive all states in a rectangle to the adjacent facet with the next rectangle. The overall control law can be easily rendered continuous, by making sure that on the common facet, in both adjacent rectangles the same choice is made for the control at the vertices. Indeed, the common facet is a $N-1$-dimensional rectangle and the control law everywhere on the facet is a convex combination of the 2^{N-1} vertices of the facet.

Remark 4: The controllability properties of (11), as far as Problems 1 and 2 are concerned, are buried in (23), which not only capture rank-type conditions on B, but also physical bounds on controls.

IV. Example: Aircraft Angular Velocity Control

In this section we show how to control the angular velocity of an aircraft by using a multi-affine control system and a rectangular partition of the velocity space dictated by the task. Consider a parallelepiped aircraft of mass m with jet-gas actuators. Consider a frame $\{M\}$ fixed at the centroid of the aircraft and aligned with the principal axis, so that $G = \mathrm{diag}\{g_1, g_2, g_3\}$. Assume that $\mathrm{dimspan}\{\zeta_1,\ldots,\zeta_m\} = 3$, *i.e.,* the system is controllable. Without loss of generality, we will take the control directions as being the Euclidean basis vectors e_i, $i = 1,2,3$ and the control will be reparameterized by u_i along these directions. Then, the angular control system (2) takes the form of the known controlled Euler's equations:

$$\begin{array}{rcl} \dot{\omega}_1 & = & \frac{g_2-g_3}{g_1}\omega_2\omega_3 + u_1 \\ \dot{\omega}_2 & = & \frac{g_3-g_1}{g_2}\omega_1\omega_3 + u_2 \\ \dot{\omega}_3 & = & \frac{g_1-g_2}{g_3}\omega_1\omega_2 + u_3 \end{array} \tag{24}$$

Assuming that the aircraft spans between a_i and b_i along the direction e_i $(i = 1,2,3)$ of the body frame $\{M\}$, then we have

$$\begin{array}{l} g_1 = \frac{1}{24}m((b_2-a_2)^2 + (b_3-a_3)^2), \\ g_2 = \frac{1}{24}m((b_3-a_3)^2 + (b_1-a_1)^2), \\ g_3 = \frac{1}{24}m((b_1-a_1)^2 + (b_2-a_2)^2). \end{array} \tag{25}$$

Finally, the controls u_i are limited to take values in $[-1,1]$. The control system (24) is obviously of the form (11) with $x = \omega$, the multi-affine drift $f(x) = (x_2x_3(g_2-g_3)/g_1, x_1x_3(g_3-g_1)/g_2, x_1x_2(g_1-g_2)/g_3)$, control directions $B = I_3$, and set of admissible controls $U = [-1,1]^3$.

Consider the following control scenario. Assume that the aircraft is initially rotating around the z-axis of its body frame $\{M\}$ at speed ω_s. The goal is to control the aircraft so that it eventually rotates around its x-axis at the same speed and remains in this state for all times. Moreover, while transiting from the initial to the final states, the aircraft is forbidden to develop rotational speed ω_2 around its y-axis.

To capture the uncertainty on knowledge of the state as well as sensor noise, we allow for deviations of amplitude $\epsilon > 0$ in all directions. Under this assumption, the initial state of rotation is assumed to be the collection of all states in a small cube centered at $\omega = (0,\ 0,\ \omega_s)$ and with side 2ϵ. The amount of allowed speed of rotation around the y-axis is assumed to be ϵ and the goal is to drive and keep the system in a small cube centered at $\omega = (\omega_s,\ 0,\ 0)$, where $\epsilon > 0$ is a small number. Using the procedure presented in Section III, we can provide a solution to this problem in terms of a feedback control law by defining a set of rectangles in the velocity space and solve a control problem of the type 1.

Explicitly, according to the specifications of the task, consider a set of five pairwise adjacent rectangles. The task is accomplished if the following controllers (solutions of Problems of type 2) are designed:

- Controller 1 - "drive" the system down along the ω_3-axis from $\omega_3 = \omega_3^s + \epsilon$ to $\omega_3 = \epsilon$ while keeping the absolute values of ω_1 and ω_3 less than 2ϵ. The solution to this problem is found by applying Theorem 1 to the rectangle $[-\epsilon, \epsilon] \times [-\epsilon, \epsilon] \times [\epsilon, \omega_s + \epsilon]$ with exit facet $\omega_3 = \epsilon$.
- Controller 2 - "take the turn" around origin. This control law can be derived by applying Theorem 1 to the rectangle $[-\epsilon, \epsilon] \times [-\epsilon, \epsilon] \times [-\epsilon, \epsilon]$ with exit facet $\omega_1 = \epsilon$.
- Controller 3 - drive the system along the ω_1-axis from $\omega_1 = \epsilon$ to $\omega_1 = \omega_1^s - \epsilon$ while keeping the absolute values of ω_2 and ω_3 less than 2ϵ. The solution is found by applying Theorem 1 to the rectangle $[\epsilon, \omega_s - \epsilon] \times [-\epsilon, \epsilon] \times [-\epsilon, \epsilon]$ with exit facet $\omega_1 = \omega_s - \epsilon$.
- Controller 4 - keeps the system in a cubic box centered at $(\omega_3^s, 0, 0)$ and side 2ϵ. The controller is designed by applying Corollary 1 to the rectangle $[\omega_s - \epsilon, \omega_s + \epsilon] \times [-\epsilon, \epsilon] \times [-\epsilon + \epsilon]$.

We used the following numerical data:

$$\omega_s = 3,\ m = 1,\ a_1 = -4,\ b_1 = 4,\ a_2 = -4,\\ b_2 = 4,\ a_3 = -1,\ b_3 = 1,\ \epsilon = 0.1.$$

A possible choice of Controllers 1, 2, 3, and 4 is given below. For each of the rectangles, the controls at the vertices were obtained as solutions of the set of linear inequalities (23) intersected with the admissible set U. u^i, $i = 1, \ldots, 4$ is the feedback control valid everywhere in the corresponding rectangle, uniquely determined by its values at the vertices.

a) Controller 1:

$$u^1(x) = \begin{bmatrix} -2.5x_1 + 2.5x_2 \\ -2.5x_1 - 2.5x_2 \\ -0.5 \end{bmatrix}$$

b) Controller 2:

$$u^2(x) = \begin{bmatrix} 0.25 + 25x_2x_3 \\ -25x_1x_3 - 2.5x_2 \\ -5x_3 \end{bmatrix}$$

c) Controller 3:

$$u^3(x) = \begin{bmatrix} 0.25 + 25x_2x_3 \\ -2.5x_2 - 2.5x_3 \\ -5x_3 \end{bmatrix}$$

d) Controller 4:

$$u^4(x) = \begin{bmatrix} 7.5 - 2.5x_1 + 25x_2x_3 \\ -2.5x_2 - 2.5x_3 \\ -5x_3 \end{bmatrix}$$

A closed loop system trajectory in the velocity space starting from $(0, 0, 3)$ is shown for illustration in Figure 1. It can be seen that all the specifications are satisfied. Figure 1 (b) shows the amounts of time spent by the system in each of the rectangles. The values on the y-axis correspond to controller index (1,2,3,4).

V. Conclusion

In this paper we approach the problem of controlling gas-jet aircraft and underwater vehicles from a formal analysis perspective. We start from the observation that these control systems are of a particular form, which allows the application of a powerful and computationally efficient algorithm to automatically generate feedback controls to drive all the states from an initial region to a final region of the state space. This method can be used for repositioning or changing the velocity profile under control and state constraints. If stabilization to a particular state inside the final region is desired, then one of the many locally stabilizing controllers presented in literature can be used. An example of controlling the angular velocity of a gas-jet aircraft is presented for illustration.

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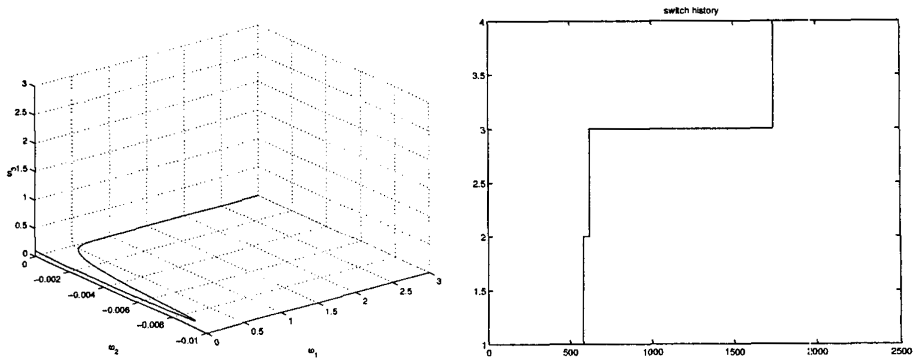

(a) (b)

Fig. 1. (a) Trajectory in the velocity space, (b) The mode switch history.

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