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Sufficient conditions for feasibility of optimal control problems using Control Barrier Functions ☆

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ABSTRACT

It has been shown that satisfying state and control constraints while optimizing quadratic costs subject to desired (sets of) state convergence for affine control systems can be reduced to a sequence of quadratic programs (QPs) by using Control Barrier Functions (CBFs) and Control Lyapunov Functions (CLFs). One of the main challenges in this approach is ensuring the feasibility of these QPs, especially under tight control bounds and safety constraints of high relative degree. The main contribution of this paper is to provide sufficient conditions for guaranteed feasibility. The sufficient conditions are captured by a single constraint that is enforced by a CBF, which is added to the QPs such that their feasibility is always guaranteed. The additional constraint is designed to be always compatible with the existing constraints, therefore, it cannot make a feasible set of constraints infeasible — it can only increase the overall feasibility. We illustrate the effectiveness of the proposed approach on an adaptive cruise control problem.

1. Introduction

Constrained optimal control problems with safety specifications are central to increasingly widespread safety critical autonomous and cyber physical systems. Traditional Hamiltonian analysis Bryson and Ho (1969) Abu-Khalaf et al. (2006) and dynamic programming Denardo (2003) cannot accommodate the size and nonlinearities of such systems, and their applicability is mostly limited to linear systems. Model Predictive Control (MPC) Rawlings et al. (2018) methods have been shown to work for large, non-linear systems. However, safety requirements are hard to be guaranteed between time intervals in MPC. Motivated by these limitations, barrier and control barrier functions enforcing safety have received increased attention in the past years Ames et al. (2014), Glotfelter et al. (2017) and Xiao and Belta (2019).

Barrier functions (BFs) are Lyapunov-like functions Tee et al. (2009), Wieland and Allgower (2007), whose use can be traced back to optimization problems Boyd and Vandenberghe (2004). More recently, they have been employed to prove set invariance Aubin (2009), Prajna et al. (2007) and Wisniewski and Sloth (2013) and to address multi-objective control problems Panagou et al. (2013). In Tee et al. (2009), it was proved that if a BF for a given set satisfies Lyapunov-like conditions, then the set is forward invariant. A less restrictive form of a BF, which is allowed to grow when far away from the boundary of the set, was proposed in Ames et al. (2014). Another approach that allows a BF to be zero was proposed in Glotfelter et al. (2017), Lindemann and Dimarogonas (2019). This simpler form has also been considered in time-varying cases and applied to enforce Signal Temporal Logic (STL) formulas as hard constraints Lindemann and Dimarogonas (2019).

Control BFs (CBFs) are extensions of BFs for control systems, and are used to map a constraint defined over system states to a constraint on the control input. The CBFs from Ames et al. (2014) and Glotfelter et al. (2017) work for constraints that have relative degree one with respect to the system dynamics. A backstepping approach was introduced in Hsu et al. (2015) to address higher relative degree constraints, and it was shown to work for relative degree two. A CBF method for position-based constraints with relative degree two was also proposed in Wu and Sreenath (2015). A more general form was considered in Nguyen and Sreenath (2016), which works for arbitrarily high relative degree constraints, employs input–output linearization and finds a pole placement controller with negative poles to stabilize an exponential CBF to zero. The high order CBF (HOCBF) proposed in Xiao and Belta (2019) is simpler and more general than the exponential CBF Nguyen and Sreenath (2016).

Most works using CBFs to enforce safety are based on the assumption that the (nonlinear) control system is affine in controls and the cost is quadratic in controls. Convergence to desired states is achieved by using Control Lyapunov Functions

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(CLFs) Ames et al. (2012). The time domain is discretized, and the state is assumed to be constant within each time step (at its value at the beginning of the step). The optimal control problem becomes a Quadratic Program (QP) in each time step, and the optimal control value is kept constant over each such step. Using this approach, the original optimal control problem is reduced to a (possibly large) sequence of quadratic programs (QP) — one for each interval Galloway et al. (2015). While computationally efficient, this myopic approach can easily lead to infeasibility: the constant optimal control derived at the beginning of an interval can lead the system to a state that gives incompatible control constraints at the end of the interval, rendering the QP corresponding to the next time interval infeasible.

For the particular case of an adaptive cruise control (ACC) problem in Ames et al. (2014), it was shown that an additional constraint (minimum braking distance) can help keep the system away from states leading to incompatibility of control CBF and CLF constraints. However, this additional constraint itself may conflict with other constraints in the ACC problem, such as the control bounds. To guarantee the problem feasibility for more general optimal control problems with the CBF method, the penalty method Xiao and Belta (2019) and adaptive CBF Xiao et al. (2021a) were proposed; however, these two approaches are case-dependent and often studied under worst-case conditions. Moreover, they are not analytical approaches (i.e., no closed-form solutions are derived and numerical techniques are required to tune the penalties) making them hard to further study system performance for general constrained optimal control problems.

The main contribution of this paper is to provide a novel method to find sufficient conditions to guarantee the feasibility of CBF-CLF based QPs. To the best of our knowledge, this is the first paper in the literature that provides sufficient conditions to guarantee the feasibility of these QPs. This is achieved by the proposed feasibility constraint method that makes the problem constraints compatible in terms of control given an arbitrary system state. The sufficient conditions are captured by a single constraint that is enforced by a CBF, and is added to the problem to formulate the sequence of QPs mentioned above with guaranteed feasibility. The added constraint is always compatible with the existing constraints and, therefore, it cannot make a feasible set of constraints infeasible. However, by "shaping" the constraint set of a current QP, it guarantees the feasibility of the next QP in the sequence. We illustrate our approach and compare it to other methods on an ACC problem.

The remainder of the paper is organized as follows. In Section 2, we provide preliminaries on HOCBF and CLF. Section 3 formulates an optimal control problem and outlines our CBF-based solution approach. We show how we can find a feasibility constraint for an optimal control problem in Section 4, and present case studies and simulation results in Section 5. We conclude the paper in Section 6.

2. Preliminaries

Definition 1 (*Class $\mathcal{K}$ Function Khalil, 2002*)**.** A continuous function $\alpha : [0, a) \to [0, \infty), a > 0$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.

Consider an affine control system of the form

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} \tag{1}$$

where $\boldsymbol{x} \in X \subset \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times q}$ are locally Lipschitz, and $\boldsymbol{u} \in U \subset \mathbb{R}^q$ is the control constraint set defined as

$$U := \{\boldsymbol{u} \in \mathbb{R}^q : \boldsymbol{u}_{min} \le \boldsymbol{u} \le \boldsymbol{u}_{max}\}. \tag{2}$$

with $\boldsymbol{u}_{min}, \boldsymbol{u}_{max} \in \mathbb{R}^q$ and the inequalities are interpreted componentwise.

Definition 2. A set $C \subset \mathbb{R}^n$ is forward invariant for system (1) if its solutions starting at any $\boldsymbol{x}(0) \in C$ satisfy $\boldsymbol{x}(t) \in C$, $\forall t \ge 0$.

Definition 3 (*Relative Degree*)**.** The relative degree of a (sufficiently many times) differentiable function $b : \mathbb{R}^n \to \mathbb{R}$ with respect to system (1) is the number of times it needs to be differentiated along its dynamics until the control $\boldsymbol{u}$ explicitly shows in the corresponding derivative.

In this paper, since function b is used to define a constraint $b(\boldsymbol{x}) \ge 0$, we will also refer to the relative degree of b as the relative degree of the constraint.

For a constraint $b(\boldsymbol{x}) \ge 0$ with relative degree m, $b : \mathbb{R}^n \to \mathbb{R}$, and $\psi_0(\boldsymbol{x}) := b(\boldsymbol{x})$, we define a sequence of functions $\psi_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, \dots, m\}$:

$$\psi_i(\boldsymbol{x}) := \dot{\psi}_{i-1}(\boldsymbol{x}) + \alpha_i(\psi_{i-1}(\boldsymbol{x})), i \in \{1, \dots, m\}, \tag{3}$$

where $\alpha_i(\cdot)$, $i \in \{1, \dots, m\}$ denotes a $(m-i)^{th}$ order differentiable class $\mathcal{K}$ function.

We further define a sequence of sets C_i, $i \in \{1, \dots, m\}$ associated with (3) in the form:

$$C_i := \{\boldsymbol{x} \in \mathbb{R}^n : \psi_{i-1}(\boldsymbol{x}) \ge 0\}, i \in \{1, \dots, m\}. \tag{4}$$

Definition 4 (*High Order Control Barrier Function (HOCBF) Xiao & Belta, 2019*)**.** Let $C_1, \dots, C_m$ be defined by (4) and $\psi_1(\boldsymbol{x}), \dots, \psi_m(\boldsymbol{x})$ be defined by (3). A function $b : \mathbb{R}^n \to \mathbb{R}$ is a High Order Control Barrier Function (HOCBF) of relative degree m for system (1) if there exist $(m-i)^{th}$ order differentiable class $\mathcal{K}$ functions α_i, $i \in \{1, \dots, m-1\}$ and a class $\mathcal{K}$ function α_m such that

$$\sup_{\boldsymbol{u}\in U}[L_f^m b(\boldsymbol{x}) + L_g L_f^{m-1} b(\boldsymbol{x})\boldsymbol{u} + S(b(\boldsymbol{x})) + \alpha_m(\psi_{m-1}(\boldsymbol{x}))] \ge 0, \tag{5}$$

for all $\boldsymbol{x} \in C_1 \cap, \dots, \cap C_m$. In (5), L_f and L_g denote the Lie derivatives along f and g, respectively, L_f^m denotes Lie derivatives along f m times, and $S(\cdot)$ denotes the remaining Lie derivatives along f with degree less than or equal to $m-1$ (omitted for simplicity, see Xiao et al., 2021a).

The HOCBF is a general form of the relative degree one CBF Ames et al. (2014), Glotfelter et al. (2017), Lindemann and Dimarogonas (2019) (setting $m = 1$ reduces the HOCBF to the common CBF form in Ames et al., 2014, Glotfelter et al., 2017, Lindemann & Dimarogonas, 2019), and it is also a general form of the exponential CBF Nguyen and Sreenath (2016).

Theorem 1 (*Xiao & Belta, 2019*)**.** *Given a HOCBF $b(\boldsymbol{x})$ from Definition 4 with the associated sets $C_1, \dots, C_m$ defined by (4), if $\boldsymbol{x}(0) \in C_1 \cap, \dots, \cap C_m$, then any Lipschitz continuous controller $\boldsymbol{u}(t)$ that satisfies (5), $\forall t \ge 0$ renders $C_1 \cap, \dots, \cap C_m$ forward invariant for system (1).*

It is important to note that the satisfaction of the CBF (HOCBF) constraint (5) is only a sufficient condition for the satisfaction of the original constraint $b(\boldsymbol{x}) \ge 0$. This makes the existing CBF (HOCBF) method conservative and may limit the performance of the system. In order to address this conservativeness, an adaptive CBF is proposed in Xiao et al. (2021a), and its satisfaction is a necessary and sufficient condition for the satisfaction of the original constraint $b(\boldsymbol{x}) \ge 0$.

Definition 5 (*Control Lyapunov Function (CLF) Ames et al., 2012*)**.** A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is an exponentially stabilizing control Lyapunov function (CLF) for system (1) if there exist constants $c_1 > 0, c_2 > 0, c_3 > 0$ such that for all $\boldsymbol{x} \in X$, $c_1\|\boldsymbol{x}\|^2 \le V(\boldsymbol{x}) \le c_2\|\boldsymbol{x}\|^2$,

$$\inf_{\boldsymbol{u}\in U}[L_f V(\boldsymbol{x}) + L_g V(\boldsymbol{x})\boldsymbol{u} + c_3 V(\boldsymbol{x})] \le 0. \tag{6}$$

Many existing works Ames et al. (2014), Nguyen and Sreenath (2016), Yang et al. (2019) combine CBFs for systems with relative degree one with quadratic costs to form optimization problems. Time is discretized and an optimization problem with constraints given by the CBFs (inequalities of the form (5)) is solved at each time step. The inter-sampling effect is considered in Yang et al. (2019). If convergence to a state is desired, then a CLF constraint of the form (6) is added, as in Ames et al. (2014) Yang et al. (2019). Note that these constraints are linear in control since the state value is fixed at the beginning of the interval, therefore, each optimization problem is a quadratic program (QP). The optimal control obtained by solving each QP is applied at the current time step and held constant for the whole interval. The state is updated using dynamics (1), and the procedure is repeated. Replacing CBFs by HOCBFs allows us to handle constraints with arbitrary relative degree Xiao and Belta (2019). This method works conditioned on the fact that the QP at every time step is feasible. However, this is not guaranteed, in particular under tight control bounds. In this paper, we show how we can find sufficient conditions for the feasibility of the QPs.

3. Problem formulation and approach

Objective: (Minimizing cost) Consider an optimal control problem for the system in (1) with the cost defined as:

$$J(\boldsymbol{u}(t)) = \int_0^T \mathcal{C}(\|\boldsymbol{u}(t)\|)dt + p\|\boldsymbol{x}(T) - \boldsymbol{K}\|^2 \tag{7}$$

where $\|\cdot\|$ denotes the 2-norm of a vector, $\mathcal{C}(\cdot)$ is a strictly increasing function of its argument, and $T > 0, p > 0$. $\boldsymbol{K} \in \mathbb{R}^n$ is a desired state, which is assumed to be an equilibrium for the system. Associated with this problem are the requirements that follow.

Constraint1 (Safety constraints): System (1) should always satisfy one or more safety requirements of the form:

$$b(\boldsymbol{x}(t)) \geq 0, \boldsymbol{x} \in X, \forall t \in [0, T]. \tag{8}$$

where $b : \mathbb{R}^n \to \mathbb{R}$ is assumed to be continuously differentiable. If not, we may overapproximate it by some continuously differentiable constraints (e.g., using the optimal disk coverage approach introduced for autonomous driving in Xiao et al., 2021b). Moreover, when we have multiple safety constraints, we assume that they do not conflict with each other. Otherwise, we may relax some of them according to their priorities (if such are known), as shown in Xiao et al. (2021b).

Constraint2 (Control constraints): The control must satisfy (2) for all $t \in [0, T]$.

A control policy for system (1) is ***feasible*** if constraints (8) and (2) are satisfied for all times. In this paper, we consider the following problem:

Problem 1. Find a feasible control policy for system (1) such that the cost (7) is minimized.

Approach: We use a HOCBF to enforce (8), and use a relaxed CLF to achieve the convergence requirement in (7). If the cost (7) is quadratic in $\boldsymbol{u}$, then we can formalize Problem 1 using a CBF-CLF-QP approach Ames et al. (2014), with the CBF replaced by the HOCBF Xiao and Belta (2019):

$$\min_{\boldsymbol{u}(t),\delta(t)} \int_0^T \|\boldsymbol{u}(t)\|^2 + p\delta^2(t)dt \tag{9}$$

subject to

$$L_f^m b(\boldsymbol{x}) + L_g L_f^{m-1} b(\boldsymbol{x})\boldsymbol{u} + S(b(\boldsymbol{x})) + \alpha_m(\psi_{m-1}(\boldsymbol{x})) \geq 0, \tag{10}$$

$$L_f V(\boldsymbol{x}) + L_g V(\boldsymbol{x})\boldsymbol{u} + \epsilon V(\boldsymbol{x}) \leq \delta(t), \tag{11}$$

$$\boldsymbol{u}_{min} \leq \boldsymbol{u} \leq \boldsymbol{u}_{max}, \tag{12}$$

where $V(\boldsymbol{x}) = (\boldsymbol{x}(t) - \boldsymbol{K})^T P(\boldsymbol{x}(t) - \boldsymbol{K})$, P is positive definite, $c_3 = \epsilon > 0$ in Definition 5, $p > 0$, and $\delta(t)$ is a relaxation (decision variable) that we wish to minimize for the CLF constraint. We assume that $b(\boldsymbol{x})$ has relative degree m. The above optimization problem is *feasible at a given state* $\boldsymbol{x}$ if all the constraints define a non-empty set for the decision variables $\boldsymbol{u}, \delta$.

The optimal control problem (9), (10), (11), (12) with decision variables $\boldsymbol{u}(t), \delta(t)$ is usually solved point-wise, as outlined in the end of Section 2. The time interval $[0, T]$ is divided into a finite number of intervals $[t_k, t_{k+1}), k = 0, 1, 2, \dots, t_0 = 0$. At every discrete time t_k defining the bounds of the intervals, we fix the state $\boldsymbol{x}(t_k)$, so that the optimal control problem above becomes a QP:

$$(\boldsymbol{u}^*(t_k), \delta^*(t_k)) = \arg\min_{\boldsymbol{u}(t_k),\delta(t_k)} \|\boldsymbol{u}(t_k)\|^2 + p\delta^2(t_k),$$
$$\text{s.t. (10), (11), (12).}$$

We obtain an optimal control $\boldsymbol{u}^*(t_k)$ from the above QP and we apply it to system (1) for the whole interval $[t_k, t_{k+1})$. It is important to note that this approach is different from MPC as there is no receding horizon involved. The CBF method focuses on safety guarantees, and is usually based on following a given optimal reference trajectory.

This paper is motivated by the fact that this computationally efficient but myopic approach can easily lead to infeasible QPs, especially under tight control bounds. In other words, after we apply the constant $\boldsymbol{u}^*(\bar{t})$ to system (1) starting at $\boldsymbol{x}(\bar{t})$ for the whole interval that starts at $\bar{t}$, we may end up at a state where the HOCBF constraint (10) conflicts with the control bounds (12), which would render the QP corresponding to the next time interval infeasible.[1] To avoid this, we define an additional *feasibility constraint*:

Definition 6 (*Feasibility Constraint*)**.** Suppose the QP (9), subject to (10), (11) and (12), is feasible at the current state $\boldsymbol{x}(\bar{t}), \bar{t} \in [0, T)$. A constraint $b_F(\boldsymbol{x}) \geq 0$, where $b_F : \mathbb{R}^n \to \mathbb{R}$, is a feasibility constraint if it makes the QP corresponding to the next time interval feasible.

In order to ensure that the QP (9), subject to (10), (11) and (12), is feasible for the next time interval, a feasibility constraint $b_F(\boldsymbol{x}) \geq 0$ should have two important features: (i) it guarantees that (10) and (12) do not conflict, (ii) the feasibility constraint itself does not conflict with both (10) and (12) at the same time.

An illustrative example of how a feasibility constraint works is shown in Fig. 1. A robot whose control is determined by solving the QP (9), subject to (10), (11) and (12), will run close to an obstacle in the following step. The next state may be infeasible for the QP associated with that next step. For example, the state denoted by the red dot in Fig. 1 may have large speed such that the robot cannot find a control to avoid the obstacle in the next step. If a feasibility constraint can prevent the robot from reaching this state, then the QP is feasible.

After we find a feasibility constraint, we can enforce it through a CBF and take it as an additional constraint for (9) to guarantee the feasibility given system state $\boldsymbol{x}$. We show how we can determine an appropriate feasibility constraint in the following section.

4. Feasibility constraint

We begin with a simple example to illustrate the necessity for a feasibility constraint for the CBF-CLF based QPs.

[1] Note that, since the CLF constraint (11) is relaxed, it does not affect the feasibility of the QP.

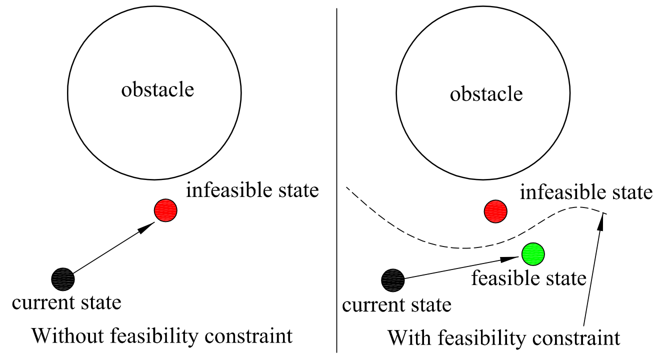

Fig. 1. An illustration of how a feasibility constraint works for a robot control problem. A feasibility constraint prevents the robot from going into the infeasible state.

4.1. Example: Adaptive cruise control

Consider the adaptive cruise control (ACC) problem with the ego (controlled) vehicle dynamics in the form:

$$\underbrace{\begin{bmatrix} \dot{v}(t) \\ \dot{z}(t) \end{bmatrix}}_{\dot{\boldsymbol{x}}(t)} = \underbrace{\begin{bmatrix} -\frac{1}{M}F_r(v(t)) \\ v_p - v(t) \end{bmatrix}}_{f(\boldsymbol{x}(t))} + \underbrace{\begin{bmatrix} \frac{1}{M} \\ 0 \end{bmatrix}}_{g(\boldsymbol{x}(t))} u(t) \tag{13}$$

where M denotes the mass of the ego vehicle, $z(t)$ denotes the distance between the preceding and the ego vehicles, $v_p \geq 0, v(t) \geq 0$ denote the speeds of the preceding and the ego vehicles, respectively, and $F_r(v(t))$ denotes the resistance force, which is expressed Khalil (2002) as:

$$F_r(v(t)) = f_0 sgn(v(t)) + f_1 v(t) + f_2 v^2(t),$$

where $f_0 > 0$, $f_1 > 0$ and $f_2 > 0$ are scalars determined empirically. The first term in $F_r(v(t))$ denotes the Coulomb friction force, the second term denotes the viscous friction force and the last term denotes the aerodynamic drag. The control $u(t)$ is the driving force of the ego vehicle subject to the constraint:

$$-c_d Mg \leq u(t) \leq c_a Mg, \forall t \geq 0, \tag{14}$$

where $c_a > 0$ and $c_d > 0$ are the maximum acceleration and deceleration coefficients, respectively, and g is the gravity constant.

We require that the distance $z(t)$ between the ego vehicle and its immediately preceding vehicle be greater than $l_0 > 0$, i.e.,

$$z(t) \geq l_0, \forall t \geq 0. \tag{15}$$

Let $b(\boldsymbol{x}(t)) := z(t) - l_0$. The relative degree of $b(\boldsymbol{x}(t))$ is $m = 2$, so we choose a HOCBF following Definition 4 by defining $\psi_0(\boldsymbol{x}(t)) := b(\boldsymbol{x}(t))$, $\alpha_1(\psi_0(\boldsymbol{x}(t))) := p_1\psi_0(\boldsymbol{x}(t))$ and $\alpha_2(\psi_1(\boldsymbol{x}(t))) := p_2\psi_1(\boldsymbol{x}(t))$, $p_1 > 0, p_2 > 0$. We then seek a control for the ego vehicle such that the constraint (15) is satisfied. The control $u(t)$ should satisfy (5) which in this case is:

$$\underbrace{\frac{F_r(v(t))}{M}}_{L_f^2 b(\boldsymbol{x}(t))} + \underbrace{\frac{-1}{M}}_{L_g L_f b(\boldsymbol{x}(t))} \times u(t) + \underbrace{p_1(v_p - v(t))}_{S(b(\boldsymbol{x}(t)))} + \underbrace{p_2(v_p - v(t)) + p_1 p_2 (z(t) - l_0)}_{\alpha_2(\psi_1(\boldsymbol{x}(t)))} \geq 0. \tag{16}$$

Suppose we wish to minimize $\int_0^T \left(\frac{u(t)-F_r(v(t))}{M}\right)^2 dt$, in which case we have a constrained optimal control problem. We can then use the QP-based method introduced at the end of the last section to solve this ACC problem. However, the HOCBF constraint (16) can easily conflict with $-c_d Mg \leq u(t)$ in (14), i.e., the ego vehicle cannot brake in time under control constraint (2) so that the safety constraint (15) is satisfied when the two vehicles get close to each other. This is intuitive when we rewrite (16) in the form:

$$\frac{1}{M}u(t) \leq \frac{F_r(v(t))}{M} + (p_1+p_2)(v_p - v(t)) + p_1 p_2 (z(t) - l_0). \tag{17}$$

The right-hand side above is usually negative when the two vehicles get close to each other. If it is smaller than $-c_d Mg$, the HOCBF constraint (16) will conflict with $-c_d Mg \leq u(t)$ in (14). When this happens, the QP will be infeasible. In the rest of the paper, we show how we can solve this infeasibility problem in general by a feasibility constraint as in Definition 6.

4.2. Feasibility constraint for relative-degree-one safety constraints

It is important to first point out that our analysis does not depend on the relative degree of the constraints. Therefore, for ease of exposition, we start with feasibility constraints for a relative-degree-one safety constraint, and then generalize it to the case of high-relative-degree safety constraints.

Suppose we have a constraint $b(\boldsymbol{x}) \geq 0$ with relative degree one for system (1), where $b : \mathbb{R}^n \to \mathbb{R}$. Then we can define $b(\boldsymbol{x})$ as a HOCBF with $m = 1$ as in Definition 4, i.e., we have a "traditional" CBF. Following (5), any control $\boldsymbol{u} \in U$ should satisfy the CBF constraint:

$$-L_g b(\boldsymbol{x})\boldsymbol{u} \leq L_f b(\boldsymbol{x}) + \alpha(b(\boldsymbol{x})), \tag{18}$$

where $\alpha(\cdot)$ is a class $\mathcal{K}$ function of its argument. We define a set of controls that satisfy the last equation as:

$$K(\boldsymbol{x}) = \{\boldsymbol{u} \in \mathbb{R}^q : -L_g b(\boldsymbol{x})\boldsymbol{u} \leq L_f b(\boldsymbol{x}) + \alpha(b(\boldsymbol{x}))\}. \tag{19}$$

Our analysis for determining a feasibility constraint depends on whether any component of the vector $L_g b(\boldsymbol{x})$ will change sign in the time interval $[0, T]$ or not.

(1) All components in $L_g b(\boldsymbol{x})$ do not change sign: Since all components in $L_g b(\boldsymbol{x})$ do not change sign for all $\boldsymbol{x} \in X$, the inequality constraint for each control component does not change sign if we multiply each component of $L_g b(\boldsymbol{x})$ by the corresponding one of the control bounds in (2). Therefore, we assume that $L_g b(\boldsymbol{x}) \leq \boldsymbol{0}$ (componentwise), $\boldsymbol{0} \in \mathbb{R}^q$ in the rest of this section. The analysis for other cases (each component of $L_g b(\boldsymbol{x})$ is either non-negative or non-positive) is similar. Not all the components in $L_g b(\boldsymbol{x})$ can be 0 due to the relative degree definition in Definition 3. We can multiply the control bounds (2) by the vector $-L_g b(\boldsymbol{x})$, and get

$$-L_g b(\boldsymbol{x})\boldsymbol{u}_{\min} \leq -L_g b(\boldsymbol{x})\boldsymbol{u} \leq -L_g b(\boldsymbol{x})\boldsymbol{u}_{\max}, \tag{20}$$

The control constraint (20) is actually a relaxation of the control bound (2) as we multiply each component of $L_g b(\boldsymbol{x})$ by the corresponding one of the control bounds in (2), and then add them together. We define

$$U_{ex}(\boldsymbol{x}) = \{\boldsymbol{u} \in \mathbb{R}^q : -L_g b(\boldsymbol{x})\boldsymbol{u}_{\min} \leq -L_g b(\boldsymbol{x})\boldsymbol{u} \leq -L_g b(\boldsymbol{x})\boldsymbol{u}_{\max}\}, \tag{21}$$

We also provide the following formal definition describing how two or more state-dependent control constraints are "conflict-free":

Definition 7 (*Conflict-free*)**.** We define two (or more) state-dependent control constraints to be conflict-free if the intersection of the two (or more) sets defined by these constraints in terms of $\boldsymbol{u}$ are non-empty for all $\boldsymbol{x} \in X$.

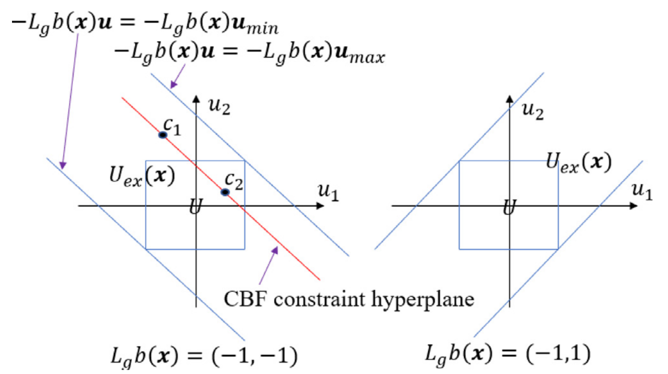

Fig. 2. The relationship between $U \subset U_{ex}(\boldsymbol{x})$ and $U_{ex}(\boldsymbol{x})$ in the case of a two-dimensional control $\boldsymbol{u} = (u_1, u_2)$. The magnitude of $L_g b(\boldsymbol{x})$ determines the slopes of the two lines (hyperplanes) $-L_g b(\boldsymbol{x})\boldsymbol{u} = -L_g b(\boldsymbol{x})\boldsymbol{u}_{max}$ and $-L_g b(\boldsymbol{x})\boldsymbol{u} = -L_g b(\boldsymbol{x})\boldsymbol{u}_{min}$. If there exists a control $c_1 \in U_{ex}(\boldsymbol{x})$ that satisfies the CBF constraint (18) (on the boundary), then there exists a control $c_2 \in U$ that also satisfies the CBF constraint (18) (on the boundary).

It is obvious that U is a subset of $U_{ex}(\boldsymbol{x})$. An example of a two-dimensional control $\boldsymbol{u} = (u_1, u_2)$ is shown in Fig. 2. Nonetheless, the relaxation set $U_{ex}(\boldsymbol{x})$ does not negatively affect the property of the following lemma:

Lemma 1. *If the control $\boldsymbol{u}$ is such that (20) is conflict-free with (18) for all $\boldsymbol{x} \in X$, then the control bound (2) is also conflict-free with (18).*

Proof. Let $g = (g_1, \dots, g_q)$ in (1), where $g_i : \mathbb{R}^n \to \mathbb{R}^n, i, \in \{1, \dots, q\}$. We have that $L_g b(\boldsymbol{x}) = (L_{g_1} b(\boldsymbol{x}), \dots, L_{g_q} b(\boldsymbol{x})) \in \mathbb{R}^{1\times q}$. For the control bound $u_{i,min} \le u_i \le u_{i,max}, i \in \{1, \dots, q\}$ in (2), we can multiply by $-L_{g_i} b(\boldsymbol{x})$ and get

$$-L_{g_i} b(\boldsymbol{x}) u_{i,min} \le -L_{g_i} b(\boldsymbol{x}) u_i \le -L_{g_i} b(\boldsymbol{x}) u_{i,max}, \quad i \in \{1, \dots, q\},$$

as we have assumed that $L_g b(\boldsymbol{x}) \le \mathbf{0}$. If we take the summation of the inequality above over all $i \in \{1, \dots, q\}$, then we obtain the constraint (20). Therefore, the satisfaction of (2) implies the satisfaction of (20). Then U defined in (2) is a subset of $U_{ex}(\boldsymbol{x})$. It is obvious that the boundaries of the set $U_{ex}(\boldsymbol{x})$ in (21) and $K(\boldsymbol{x})$ in (19) are hyperplanes, and these boundaries are parallel to each other for all $\boldsymbol{x} \in X$. Meanwhile, the two boundaries of $U_{ex}(\boldsymbol{x})$ pass through the two corners $\boldsymbol{u}_{\min}, \boldsymbol{u}_{\max}$ of the set U (a polyhedron) following (21), respectively. If there exists a control $c_1 \in U_{ex}(\boldsymbol{x})$ (e.g., in Fig. 2) that satisfies (18), then the boundary of the set $K(\boldsymbol{x})$ in (19) lies either between the two hyperplanes defined by $U_{ex}(\boldsymbol{x})$ or above these two hyperplanes (i.e., $U_{ex}(\boldsymbol{x})$ is a subset of $K(\boldsymbol{x})$ in (19)). In the latter case, this lemma is true as U is a subset of $U_{ex}(\boldsymbol{x})$. In the former case, we can always find another control $c_2 \in U$ (e.g., in Fig. 2) that satisfies (18) as the boundary of $K(\boldsymbol{x})$ in (19) is parallel to the two $U_{ex}(\boldsymbol{x})$ boundaries that respectively pass through the two corners $\boldsymbol{u}_{\min}, \boldsymbol{u}_{\max}$ of the set U. Therefore, although U is a subset of $U_{ex}(\boldsymbol{x})$, it follows that if (20) is conflict-free with (18) in terms of $\boldsymbol{u}$ for all $\boldsymbol{x} \in X$, the control bound (2) is also conflict-free with (18). ■

Motivated by Lemma 1, in order to determine if (18) complies with (2), we may just consider (18) and (20). Since there are two inequalities in (20), we have two cases to consider: $(i) -L_g b(\boldsymbol{x})\boldsymbol{u} \le -L_g b(\boldsymbol{x})\boldsymbol{u}_{\max}$ and (18); $(ii) -L_g b(\boldsymbol{x})\boldsymbol{u}_{\min} \le -L_g b(\boldsymbol{x})\boldsymbol{u}$ and (18). It is obvious that there always exists a control $\boldsymbol{u}$ such that the two inequalities in case (i) are satisfied for all $\boldsymbol{x} \in X$, while this may not be true for case (ii), depending on $\boldsymbol{x}$. For example, the CBF for the rear-end safety constraint (15) in the ACC may conflict with the maximum braking force $-c_d M g < 0$, and it will never conflict with the maximum driving force $c_a M g > 0$ as the ego vehicle needs to brake when it gets close to the preceding vehicle in order to satisfy the safety constraint (15). Therefore, in terms of avoiding the conflict between the CBF constraint (18) and (20) that leads to the infeasibility of problem (9), subject to (10)–(12), we wish to satisfy:

$$L_f b(\boldsymbol{x}) + \alpha(b(\boldsymbol{x})) \ge -L_g b(\boldsymbol{x})\boldsymbol{u}_{\min}. \tag{22}$$

This is called the **feasibility constraint** for problem (9), subject to (10)–(12) in the case of a relative-degree-one safety constraint $b(\boldsymbol{x}) \ge 0$ in (8).

The relative degree of the feasibility constraint (22) is also one with respect to dynamics (1) as we have $b(\boldsymbol{x})$ in it. In order to find a control such that the feasibility constraint (22) is guaranteed to be satisfied, we define

$$b_F(\boldsymbol{x}) = L_f b(\boldsymbol{x}) + \alpha(b(\boldsymbol{x})) + L_g b(\boldsymbol{x})\boldsymbol{u}_{\min} \ge 0, \tag{23}$$

so that $b_F(\boldsymbol{x})$ is a CBF as in Definition 4. Then, we can get a feedback controller $K_F(\boldsymbol{x})$ that guarantees the CBF constraint (18) and the control bounds (2) do not conflict with each other:

$$K_F(\boldsymbol{x}) = \{\boldsymbol{u} \in \mathbb{R}^q : L_f b_F(\boldsymbol{x}) + L_g b_F(\boldsymbol{x})\boldsymbol{u} + \alpha_f(b_F(\boldsymbol{x})) \ge 0\}, \tag{24}$$

if $b_F(\boldsymbol{x}(0)) \ge 0$, where $\alpha_f(\cdot)$ is a class $\mathcal{K}$ function.

Theorem 2. *If Problem 1 is initially feasible and the CBF constraint in (24) corresponding to (22) does not conflict with both the control bounds (2)* and *(18) at the same time, any controller $\boldsymbol{u} \in K_F(\boldsymbol{x})$ guarantees the feasibility of problem (9), subject to (10)–(12).*

Proof. If Problem 1 is initially feasible, then the CBF constraint (18) for the safety requirement (8) does not conflict with the control bounds (2) at time 0. It also does not conflict with the constraint (20) as U is a subset of $U_{ex}(\boldsymbol{x})$ that is defined in (21). In other words, $b_F(\boldsymbol{x}(0)) \ge 0$ holds in the feasibility constraint (22). Thus, the initial condition for the CBF in Definition 4 is satisfied. By Theorem 1, we have that $b_F(\boldsymbol{x}(t)) \ge 0, \forall t \ge 0$. Therefore, the CBF constraint (18) does not conflict with the constraint (20) for all $t \ge 0$. By Lemma 1, the CBF constraint (18) also does not conflict with the control bound (2). Finally, since the CBF constraint in (24) corresponding to (22) does not conflict with the control bounds (2) and (18) at the same time by assumption, we conclude that the feasibility of the problem is guaranteed. ■

The condition "the CBF constraint in (24) corresponding to (22) does not conflict with both the control bounds (2) and (18) at the same time" in Theorem 2 is too strong. If this condition is not satisfied, then the problem can still be infeasible. In order to relax this condition, one option is to recursively define other new feasibility constraints for the feasibility constraint (22) to address the possible conflict between (24) and (2), and (18). However, the number of iterations is not bounded, and we may have a large (unbounded) set of feasibility constraints.

In order to address the unbounded iteration issue in finding feasibility constraints, we can try to express the feasibility constraint in (24) so that it is in a form which is similar to that of the CBF constraint (18). If this is achieved, we can make these two constraints compliant with each other, and thus address the unbounded iteration issue mentioned above. Therefore, we try to construct the CBF constraint in (24) so that it takes the form:

$$L_f b(\boldsymbol{x}) + L_g b(\boldsymbol{x})\boldsymbol{u} + \alpha(b(\boldsymbol{x})) + \varphi(\boldsymbol{x}, \boldsymbol{u}) \ge 0 \tag{25}$$

for some appropriately selected function $\varphi(\boldsymbol{x}, \boldsymbol{u})$. One obvious choice for $\varphi(\boldsymbol{x}, \boldsymbol{u})$ immediately following (24) is $\varphi(\boldsymbol{x}, \boldsymbol{u}) = L_f b_F(\boldsymbol{x}) + L_g b_F(\boldsymbol{x})\boldsymbol{u} + \alpha_f(b_F(\boldsymbol{x})) - L_f b(\boldsymbol{x}) - L_g b(\boldsymbol{x})\boldsymbol{u} - \alpha(b(\boldsymbol{x}))$, which can be simplified through a proper choice of the class $\mathcal{K}$ functions $\alpha(\cdot), \alpha_f(\cdot)$, as will be shown next. Since we will eventually include the constraint $\varphi(\boldsymbol{x}, \boldsymbol{u}) \geq 0$ into our QPs (shown later) to address the infeasibility problem, we wish its relative degree to be low. Otherwise, it becomes necessary to use HOCBFs to make the control show up in enforcing $\varphi(\boldsymbol{x}) \geq 0$ (instead of $\varphi(\boldsymbol{x}, \boldsymbol{u}) \geq 0$ due to its high relative degree), which could make the corresponding HOCBF constraint complicated, and make it easily conflict with the control bound (2) and the CBF constraint (18), and thus leading to the infeasibility of the QPs. Therefore, we define a candidate function as follows (note that a relative-degree-zero function means that the control $\boldsymbol{u}$ directly shows up in the function itself):

Definition 8 (*Candidate $\varphi(\boldsymbol{x}, \boldsymbol{u})$ Function*)**.** A function $\varphi(\boldsymbol{x}, \boldsymbol{u})$ in (25) is a *candidate function* if its relative degree with respect to (1) is either one or zero.

Finding candidate $\varphi(\boldsymbol{x}, \boldsymbol{u})$: In order to find a candidate $\varphi(\boldsymbol{x}, \boldsymbol{u})$ from the reformulation of the CBF constraint in (24), we can properly choose the class $\mathcal{K}$ function $\alpha(\cdot)$ in (18). A typical choice for $\alpha(\cdot)$ is a linear function, in which case we automatically have the constraint formulation (25) by substituting the function $b_F(\boldsymbol{x})$ from (23) into (24), and get

$$\begin{aligned}\varphi(\boldsymbol{x}, \boldsymbol{u}) = L_f^2 b(\boldsymbol{x}) + L_g L_f b(\boldsymbol{x})\boldsymbol{u} + L_f(L_g b(\boldsymbol{x})\boldsymbol{u}_{min})\\ + L_g(L_g b(\boldsymbol{x})\boldsymbol{u}_{min})\boldsymbol{u} + \alpha_f(b_F(\boldsymbol{x})) - b(\boldsymbol{x}).\end{aligned}$$

Note that it is possible that $L_g L_f b(\boldsymbol{x}) = 0$ and $L_g(L_g b(\boldsymbol{x})\boldsymbol{u}_{min}) = 0$ (depending on the dynamics (1) and the CBF $b(\boldsymbol{x})$), in which case the relative degree of $\varphi(\boldsymbol{x}, \boldsymbol{u})$ (written as $\varphi(\boldsymbol{x})$) is one as we have $\alpha_f(b_F(\boldsymbol{x}))$ in it and $b_F(\boldsymbol{x})$ is a function of $b(\boldsymbol{x})$.

If the relative degree of $\varphi(\boldsymbol{x}, \boldsymbol{u})$ is zero (e.g., $L_g L_f b(\boldsymbol{x}) = 0$ and $L_g(L_g b(\boldsymbol{x})\boldsymbol{u}_{min}) = 0$ are not satisfied above), we wish to require that

$$\varphi(\boldsymbol{x}, \boldsymbol{u}) \geq 0, \tag{26}$$

such that the satisfaction of the CBF constraint (18) implies the satisfaction of the CBF constraint (25), and the satisfaction of the CBF constraint (25) implies the satisfaction of (22) by Theorem 1, i.e., the CBF constraint (18) does not conflict with the control bound (2). Besides, if (26) happens to not conflict with both (18) and (2) at the same time, depending on the CBF $b(\boldsymbol{x})$ and the dynamics (1), then the QPs are guaranteed to be feasible. The CBF constraint (24) for the feasibility constraint is similar to the CBF constraint (18) for safety by properly defining the class $\mathcal{K}$ functions α, α_f, which generates (26) that needs to be satisfied. Therefore, the QP feasibility can be improved, and even be guaranteed if constraint (26) satisfies similar conditions in the following Theorem 3. This is more helpful in the case of safety constraints with high relative degree (in the next subsection) as the HOCBF constraint (5) has many complicated terms, and it is better to remove these terms in the feasibility constraint and just consider (26) in the QP in order to make (26) compliant with (18) and (2).

If the relative degree of a candidate $\varphi(\boldsymbol{x}, \boldsymbol{u})$ with respect to (1) is one, i.e., $\varphi(\boldsymbol{x}, \boldsymbol{u}) \equiv \varphi(\boldsymbol{x})$, we define a set $U_s(\boldsymbol{x})$:

$$U_s(\boldsymbol{x}) = \{\boldsymbol{u} \in \mathbb{R}^q : L_f\varphi(\boldsymbol{x}) + L_g\varphi(\boldsymbol{x})\boldsymbol{u} + \alpha_u(\varphi(\boldsymbol{x})) \geq 0\}. \tag{27}$$

where $\alpha_u(\cdot)$ is a class $\mathcal{K}$ function.

From the set of candidate functions $\varphi(\boldsymbol{x})$, if we can find one that satisfies the conditions of the following theorem, then the feasibility of problem (9), subject to (10)–(12) is guaranteed:

Theorem 3. *If $\varphi(\boldsymbol{x})$ is a candidate function such that $\varphi(\boldsymbol{x}(0)) \geq 0$, $L_f\varphi(\boldsymbol{x}) \geq 0$, $L_g\varphi(\boldsymbol{x}) = \gamma L_g b(\boldsymbol{x})$, for some $\gamma > 0, \forall \boldsymbol{x} \in X$ and $\boldsymbol{0} \in U$, then any controller $\boldsymbol{u}(t) \in U_s(\boldsymbol{x}), \forall t \geq 0$ guarantees the feasibility of problem (9), subject to (10)–(12).*

Proof. Since $\varphi(\boldsymbol{x})$ is a candidate function, we can define a set $U_s(\boldsymbol{x})$ as in (27). If $\varphi(\boldsymbol{x}(0)) \geq 0$ and $\boldsymbol{u}(t) \in U_s(\boldsymbol{x}), \forall t \geq 0$, we have that $\varphi(\boldsymbol{x}(t)) \geq 0, \forall t \geq 0$ by Theorem 1. Then, the satisfaction of the CBF constraint (18) corresponding to the safety constraint (8) implies the satisfaction of the CBF constraint (25) (equivalent to (24)) for the feasibility constraint (22). In other words, the CBF constraint (18) automatically guarantees that it will not conflict with the control constraint (20) as the satisfaction of (25) implies the satisfaction of (22) following Theorem 1 and (22) guarantees that (18) and (20) are conflict-free. By Lemma 1, the CBF constraint (18) will also not conflict with the control bound U in (2), i.e. $K(\boldsymbol{x}) \cap U \neq \emptyset$, where $K(\boldsymbol{x})$ is defined in (19).

Since $L_f\varphi(\boldsymbol{x}) \geq 0$, we have that $\boldsymbol{0} \in U_s(\boldsymbol{x})$. We also have $\boldsymbol{0} \in U(\boldsymbol{x})$, thus, $U_s(\boldsymbol{x}) \cap U \neq \emptyset$ is guaranteed. Since $L_g\varphi(\boldsymbol{x}) = \gamma L_g b(\boldsymbol{x}), \gamma > 0$, the two hyperplanes of the two half spaces formed by $U_s(\boldsymbol{x})$ in (27) and $K(\boldsymbol{x})$ in (19) are parallel to each other, and the normal directions of the two hyperplanes along the half space direction are the same. Thus, $U_s(\boldsymbol{x}) \cap K(\boldsymbol{x})$ is either $U_s(\boldsymbol{x})$ or $K(\boldsymbol{x})$, i.e., $U_s(\boldsymbol{x}) \cap K(\boldsymbol{x}) \cap U$ equals either $U_s(\boldsymbol{x}) \cap U$ or $K(\boldsymbol{x}) \cap U$. As $U_s(\boldsymbol{x}) \cap U \neq \emptyset$ and $K(\boldsymbol{x}) \cap U \neq \emptyset$, we have $U_s(\boldsymbol{x}) \cap K(\boldsymbol{x}) \cap U \neq \emptyset, \forall \boldsymbol{x} \in X$. Therefore, the CBF constraint (18) does not conflict with the control bound (2) and the CBF constraint in $U_s(\boldsymbol{x})$ at the same time, and we can conclude that the problem is guaranteed to be feasible. ■

The conditions in Theorem 3 are **sufficient conditions** for the feasibility of problem (9), subject to (10)–(12). Under the conditions in Theorem 3, we can claim that $\varphi(\boldsymbol{x}) \geq 0$ is a single **feasibility constraint** that guarantees the feasibility of problem (9), subject to (10)–(12) in the case that the safety constraint (8) is with relative degree one (i.e., $m = 1$ in (10)).

Finding valid $\varphi(\boldsymbol{x})$: A valid $\varphi(\boldsymbol{x})$ is a function that satisfies the conditions in Theorem 3. The conditions in Theorem 3 may be conservative, and how to determine such a $\varphi(\boldsymbol{x})$ function is the remaining problem. For a general system (1) and safety constraint (8), we can parameterize the definition of the CBF (18) for the safety and the CBF constraint for the feasibility constraint (24), i.e., parameterize $\alpha(\cdot)$ and $\alpha_F(\cdot)$, such as the form in Xiao et al. (2020), and then choose the parameters to satisfy the conditions in Theorem 3.

Remark 1. An example for determining such a $\varphi(\boldsymbol{x})$ for the ACC problem in Section 4.1 can be found in the end of this section. However, it is still not guaranteed that such $\varphi(\boldsymbol{x})$ functions can be found. To address this, we may consider a special class of dynamics (1), and then formulate a systematic way to derive such $\varphi(\boldsymbol{x})$ functions. In the case of such dynamics, we may even relax some of the conditions in Theorem 3. For example, if both the dynamics (1) and the safety constraint (8) are in linear forms, then the condition $L_g\varphi(\boldsymbol{x}) = \gamma L_g b(\boldsymbol{x})$, for some $\gamma > 0$ in Theorem 3 is satisfied, and thus this condition is removed.

We can now get a feasible problem from the original problem (9), subject to (10)–(12) in the form:

$$\min_{\boldsymbol{u}(t),\delta(t)} \int_0^T \|\boldsymbol{u}(t)\|^2 + p\delta^2(t)dt \tag{28}$$

subject to the feasibility constraint (26) if the relative degree of $\varphi(\boldsymbol{x}, \boldsymbol{u})$ is 0; otherwise, subject to the CBF constraint in (27). The cost (28) is also subject to the CBF constraint (18), the control bound (2), and the CLF constraint:

$$L_fV(\boldsymbol{x}) + L_gV(\boldsymbol{x})\boldsymbol{u} + \epsilon V(\boldsymbol{x}) \leq \delta(t), \tag{29}$$

where $\varphi(\boldsymbol{x})$ satisfies the conditions in Theorem 3 for (27), and (26) is assumed to be non-conflicting with the CBF constraint (18) and the control bound (2) at the same time. In order to guarantee feasibility, we may try to find a $\varphi(\boldsymbol{x})$ that has relative degree

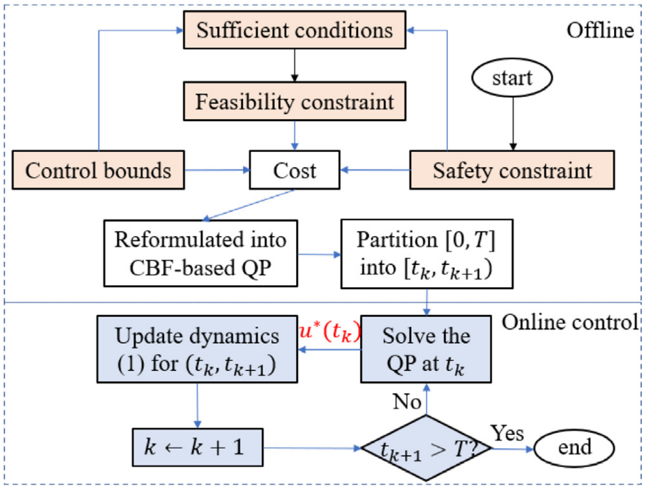

Fig. 3. The overall process of solving the constrained optimal control problem with the proposed feasibility guaranteed CBF method.

one, and that satisfies the conditions in Theorem 3. The overall process of solving the constrained optimal control problem with the feasibility guaranteed CBF method is shown in Fig. 3.

(2) Some Components in $L_g b(\boldsymbol{x})$ Change Sign: Recall that $L_g b(\boldsymbol{x}) = (L_{g_1} b(\boldsymbol{x}), \dots, L_{g_q} b(\boldsymbol{x})) \in \mathbb{R}^{1\times q}$. If $L_{g_i} b(\boldsymbol{x}), i \in \{1, \dots, q\}$ changes sign in $[0, T]$, then we have the following symmetric and non-symmetric cases to consider in order to find a valid feasibility constraint.

Let $\boldsymbol{u} = (u_1, \dots, u_q)$, $\boldsymbol{u}_{\min} = (u_{1,\min}, \dots, u_{q,\min}) \leq \boldsymbol{0}$, $\boldsymbol{u}_{\max} = (u_{1,\max}, \dots, u_{q,\max}) \geq \boldsymbol{0}$, $\boldsymbol{0} \in \mathbb{R}^q$.

Case 1: the control bound for $u_i, i \in \{1, \dots, q\}$ is symmetric, i.e. $u_{i,\max} = -u_{i,\min}$. In this case, by multiplying $-L_{g_i} b(\boldsymbol{x})$ by the control bound for u_i, we have

$$-L_{g_i} b(\boldsymbol{x}) u_{i,\min} \leq -L_{g_i} b(\boldsymbol{x}) u_i \leq -L_{g_i} b(\boldsymbol{x}) u_{i,\max} \tag{30}$$

if $L_{g_i} b(\boldsymbol{x}) < 0$. When $L_{g_i} b(\boldsymbol{x})$ changes sign at some time $t_1 \in [0, T]$, then the sign of the last equation will be reversed. However, since $u_{i,\max} = -u_{i,\min}$, we have exactly the same constraint as (30), and $-L_{g_i} b(\boldsymbol{x}) u_{i,\min}$ will still be continuously differentiable when we construct the feasibility constraint as in (22). Therefore, the feasibility constraint (22) will not be affected by the sign change of $L_{g_i} b(\boldsymbol{x}), i \in \{1, \dots, q\}$.

Case 2: the control bound for $u_i, i \in \{1, \dots, q\}$ is not symmetric, i.e., $u_{i,\max} \neq -u_{i,\min}$. In this case, we can define:

$$u_{i,\lim} := \min\{|u_{i,\min}|, u_{i,\max}\} \tag{31}$$

Considering (31), we have the following constraint

$$-u_{i,\lim} \leq u_i \leq u_{i,\lim}. \tag{32}$$

The satisfaction of the last equation implies the satisfaction of $u_{i,\min} \leq u_i \leq u_{i,\max}$ in (2).

If $L_{g_i} b(\boldsymbol{x}) < 0$, we multiply the control bound by $-L_{g_i} b(\boldsymbol{x})$ for u_i and have the following constraint

$$L_{g_i} b(\boldsymbol{x}) u_{i,\lim} \leq -L_{g_i} b(\boldsymbol{x}) u_i \leq -L_{g_i} b(\boldsymbol{x}) u_{i,\lim} \tag{33}$$

The satisfaction of (33) implies the satisfaction of (30) following (31). Now, the control bound for u_i is converted to the symmetric case, and the feasibility constraint (22) will not be affected by the sign change of $L_{g_i} b(\boldsymbol{x}), i \in \{1, \dots, q\}$.

4.3. Feasibility constraint for high-relative-degree safety constraints

Suppose we have a constraint $b(\boldsymbol{x}) \geq 0$ with relative degree $m \geq 1$ for system (1), where $b : \mathbb{R}^n \to \mathbb{R}$. Then we can define $b(\boldsymbol{x})$ as a HOCBF as in Definition 4. Any control $\boldsymbol{u} \in U$ should satisfy the HOCBF constraint (5).

In this section, we also assume that $L_g L_f^{m-1} b(\boldsymbol{x}) \leq \boldsymbol{0}, \boldsymbol{0} \in \mathbb{R}^q$ and all components in $L_g L_f^{m-1} b(\boldsymbol{x})$ do not change sign in $[0, T]$. The analysis for all other cases is similar to the last subsection.

Similar to (18), we rewrite the HOCBF constraint (5) as

$$-L_g L_f^{m-1} b(\boldsymbol{x}) \boldsymbol{u} \leq L_f^m b(\boldsymbol{x}) + S(b(\boldsymbol{x})) + \alpha_m(\psi_{m-1}(\boldsymbol{x})) \tag{34}$$

We can multiply the control bounds (2) by the vector $-L_g L_f^{m-1} b(\boldsymbol{x})$:

$$\begin{aligned} -L_g L_f^{m-1} b(\boldsymbol{x}) \boldsymbol{u}_{\min} &\leq -L_g L_f^{m-1} b(\boldsymbol{x}) \boldsymbol{u} \\ &\leq -L_g L_f^{m-1} b(\boldsymbol{x}) \boldsymbol{u}_{\max}, \end{aligned} \tag{35}$$

As in (20), the last equation is also a relaxation of the original control bound (2), and Lemma 1 still applies in the high-relative-degree-constraint case.

The HOCBF constraint (34) may conflict with the left inequality of the transformed control bound (35) when its right hand side is smaller than $-L_g L_f^{m-1} b(\boldsymbol{x}) \boldsymbol{u}_{\min}$. Therefore, we wish to have

$$L_f^m b(\boldsymbol{x}) + S(b(\boldsymbol{x})) + \alpha_m(\psi_{m-1}(\boldsymbol{x})) \geq -L_g L_f^{m-1} b(\boldsymbol{x}) \boldsymbol{u}_{\min}. \tag{36}$$

This is called the **feasibility constraint** for the problem (9), subject to (10)–(12) in the case of a high-relative-degree constraint $b(\boldsymbol{x}) \geq 0$ in (8).

In order to find a control such that the feasibility constraint (22) is guaranteed to be satisfied, we define

$$\begin{aligned} b_{hF}(\boldsymbol{x}) = L_f^m b(\boldsymbol{x}) + S(b(\boldsymbol{x})) + \alpha_m(\psi_{m-1}(\boldsymbol{x})) \\ + L_g L_f^{m-1} b(\boldsymbol{x}) \boldsymbol{u}_{\min} \geq 0, \end{aligned}$$

and define $b_{hF}(\boldsymbol{x})$ to be a HOCBF as in Definition 4.

It is important to note that the relative degree of $b_{hF}(\boldsymbol{x})$ with respect to dynamics (1) is only one, as we have $\psi_{m-1}(\boldsymbol{x})$ in it. Thus, we can get a feedback controller $K_{hF}(\boldsymbol{x})$ that guarantees free conflict between the HOCBF constraint (34) and the control bounds (2):

$$\begin{aligned} K_{hF}(\boldsymbol{x}) = \{\boldsymbol{u} \in \mathbb{R}^q : L_f b_{hF}(\boldsymbol{x}) + L_g b_{hF}(\boldsymbol{x}) \boldsymbol{u} \\ + \alpha_f(b_{hF}(\boldsymbol{x})) \geq 0\}, \end{aligned} \tag{37}$$

if $b_{hF}(\boldsymbol{x}(0)) \geq 0$, where $\alpha_f(\cdot)$ is a class $\mathcal{K}$ function.

Theorem 4. *If Problem 1 is initially feasible and the CBF constraint in (37) corresponding to (36) does not conflict with control bounds (2) and (34) at the same time, any controller $\boldsymbol{u} \in K_{hf}(\boldsymbol{x})$ guarantees the feasibility of problem (9), subject to (10)–(12).*

Proof. The proof is the same as Theorem 2. ■

Similar to the motivation for the analysis of the relative degree one case, we also reformulate the constraint in (37) in the form:

$$\begin{aligned} L_f^m b(\boldsymbol{x}) + L_g L_f^{m-1} b(\boldsymbol{x}) \boldsymbol{u} + S(b(\boldsymbol{x})) + \alpha_m(\psi_{m-1}(\boldsymbol{x})) \\ + \varphi(\boldsymbol{x}, \boldsymbol{u}) \geq 0. \end{aligned} \tag{38}$$

for some appropriate $\varphi(\boldsymbol{x}, \boldsymbol{u})$. An obvious choice is $\varphi(\boldsymbol{x}, \boldsymbol{u}) = L_f b_{hF}(\boldsymbol{x}) + L_g b_{hF}(\boldsymbol{x}) \boldsymbol{u} + \alpha_f(b_{hF}(\boldsymbol{x})) - L_f^m b(\boldsymbol{x}) - L_g L_f^{m-1} b(\boldsymbol{x}) \boldsymbol{u} - S(b(\boldsymbol{x})) - \alpha_m(\psi_{m-1}(\boldsymbol{x}))$, which is a candidate function and we wish to simplify it. We define a set $U_s(\boldsymbol{x})$ similar to (27).

Similar to the last subsection, we just consider the case that the relative degree of $\varphi(\boldsymbol{x}, \boldsymbol{u})$ is one, i.e., we have $\varphi(\boldsymbol{x})$ from now on. Then, we have the following theorem to guarantee the feasibility of the problem (9), subject to (10)–(12):

Theorem 5. *If $\varphi(\boldsymbol{x})$ is a candidate function, $\varphi(\boldsymbol{x}(0)) \geq 0$, $L_f \varphi(\boldsymbol{x}) \geq 0$, $L_g \varphi(\boldsymbol{x}) = \gamma L_g L_f^{m-1} b(\boldsymbol{x})$, for some $\gamma > 0, \forall \boldsymbol{x} \in X$ and $\boldsymbol{0} \in U$, then any controller $\boldsymbol{u}(t) \in U_s(\boldsymbol{x}), \forall t \geq 0$ guarantees the feasibility of the problem (9), subject to (10)–(12).*

Proof. The proof is the same as Theorem 3. ■

The approach to find a valid $\varphi(\boldsymbol{x})$ is the same as the last subsection. The conditions in Theorem 5 are **sufficient conditions** for the feasibility of the problem (9), subject to (10)–(12). Under the conditions in Theorem 5, we can also claim that $\varphi(\boldsymbol{x}) \geq 0$ is a single **feasibility constraint** that guarantees the feasibility of the problem (9), subject to (10)–(12) in the case that the safety constraint (8) is with high relative degree. We can get a feasible problem from the original problem (9), subject to (10)–(12) in the form:

$$\min_{\boldsymbol{u}(t),\delta(t)} \int_0^T \|\boldsymbol{u}(t)\|^2 + p\delta^2(t)dt \tag{39}$$

subject to the feasibility constraint: (26) if the relative degree of $\varphi(\boldsymbol{x},\boldsymbol{u})$ is 0; otherwise, subject to the CBF constraint in (27). The cost (39) is also subject to the HOCBF constraint (5), the control bound (2), and the CLF constraint:

$$L_f V(\boldsymbol{x}) + L_g V(\boldsymbol{x})\boldsymbol{u} + \epsilon V(\boldsymbol{x}) \leq \delta(t), \tag{40}$$

where $\varphi(\boldsymbol{x})$ satisfies the conditions in Theorem 5 for (27), and (26) is assumed to be non-conflicting with the HOCBF constraint (5) and the control bound (2) at the same time.

Remark 2. When we have multiple safety constraints, we can employ similar ideas to find sufficient conditions to guarantee problem feasibility. However, we also need to make sure that these sufficient conditions do not conflict with each other.

Example revisited. We consider the example discussed in the beginning of this section, and demonstrate how we can find a single feasibility constraint $\varphi(\boldsymbol{x}(t)) \geq 0$ for the ACC problem. It is obvious that $L_gL_fb(\boldsymbol{x}(t)) = -\frac{1}{M}$ in (16) does not change sign. The transformed control bound as in (35) for (14) is

$$-c_dg \leq \frac{1}{M}u(t) \leq c_ag. \tag{41}$$

The rewritten HOCBF constraint (17) can only conflict with the left inequality of (41). Thus, following (36) and combining (17) with (41), the feasibility constraint is $b_{hF}(\boldsymbol{x}(t)) \geq 0$, where

$$\begin{aligned} b_{hF}(\boldsymbol{x}(t)) = \frac{F_r(v(t))}{M} &+ 2(p_1+p_2)(v_p - v(t)) \\ &+ p_1p_2(z(t)-l_0) + c_dg. \end{aligned} \tag{42}$$

Since $\frac{F_r(v(t))}{M} \geq 0, \forall t \geq 0$, we can replace the last equation by

$$\begin{aligned} \hat{b}_{hF}(\boldsymbol{x}(t)) = 2(p_1+p_2)(v_p - v(t)) \\ + p_1p_2(z(t)-l_0) + c_dg. \end{aligned} \tag{43}$$

The satisfaction of $\hat{b}_{hF}(\boldsymbol{x}(t)) \geq 0$ implies the satisfaction of $b_{hF}(\boldsymbol{x}(t)) \geq 0$. Although the relative degree of (15) is two, the relative degree of $\hat{b}_{hF}(\boldsymbol{x}(t))$ is only one. We then define $\hat{b}_{hF}(\boldsymbol{x}(t))$ to be a CBF by choosing $\alpha_1(b(\boldsymbol{x}(t))) = kb(\boldsymbol{x}(t)), k > 0$ in Definition 4. Any control $u(t)$ should satisfy the CBF constraint (5) which in this case is

$$\begin{aligned} \frac{u(t)}{M} \leq \frac{F_r(v(t))}{M} &+ (\frac{p_1p_2}{p_1+p_2} + k)(v_p - v(t)) \\ &+ \frac{kp_1p_2}{p_1+p_2}(z(t)-l_0) + \frac{kc_dg}{p_1+p_2} \end{aligned} \tag{44}$$

In order to reformulate the last equation in the form of (38), we try to find k in the last equation. We require $\varphi(\boldsymbol{x}(t))$ to satisfy $L_g\varphi(\boldsymbol{x}(t)) \geq 0$ as shown in one of the conditions in Theorem 5, thus, we wish to exclude the term $z(t)-l_0$ in $\varphi(\boldsymbol{x}(t))$ since its derivative $v_p - v(t)$ is usually negative. By equating the coefficients of the term $z(t)-l_0$ in (44) and (17), we have

$$\frac{kp_1p_2}{p_1+p_2} = p_1p_2 \tag{45}$$

Thus, we get $k = p_1 + p_2$. By substituting k back into (44), we have

$$\begin{aligned} \frac{u(t)}{M} \leq \frac{F_r(v(t))}{M} &+ (p_1+p_2)(v_p - v(t)) \\ &+ p_1p_2(z(t)-l_0) + \varphi(\boldsymbol{x}(t)) \end{aligned} \tag{46}$$

where

$$\varphi(\boldsymbol{x}(t)) = \frac{p_1p_2}{p_1+p_2}(v_p - v(t)) + c_dg \tag{47}$$

It is easy to check that the relative degree of the last function is one, $L_f\varphi(\boldsymbol{x}(t)) = \frac{p_1p_2}{p_1+p_2}\frac{F_r(v(t))}{M} \geq 0$ and $L_g\varphi(\boldsymbol{x}(t)) = \frac{p_1p_2}{p_1+p_2}L_gL_fb(\boldsymbol{x}(t))$. Thus, all the conditions in Theorem 5 are satisfied except $\varphi(\boldsymbol{x}(0)) \geq 0$ which depends on the initial state $\boldsymbol{x}(0)$ of system (13). The single feasibility constraint $\varphi(\boldsymbol{x}(t)) \geq 0$ for the ACC problem is actually a speed constraint (following (47)) in this case:

$$v(t) \leq v_p + \frac{c_dg(p_1+p_2)}{p_1p_2} \tag{48}$$

If $p_1 = p_2 = 1$ in (17), we require that the half speed difference between the front and ego vehicles should be greater than $-c_dg$ in order to guarantee the ACC problem feasibility.

We can find other sufficient conditions such that the ACC problem is guaranteed to be feasible by choosing different HOCBF definitions (different class $\mathcal{K}$ functions) in the above process.

5. Case studies and simulations

In this section, we complete the ACC case study. All the computations and simulations were conducted in MATLAB. We used quadprog to solve the quadratic programs and ode45 to integrate the dynamics.

In addition to the dynamics (13), the safety constraint (15), the control bound (14), and the minimization of the cost $\int_0^T \left(\frac{u(t)-F_r(v(t))}{M}\right)^2 dt$ introduced in Section 4.1, we also consider a desired speed requirement $v \to v_d, v_d > 0$ in the ACC problem. We use the relaxed CLF as in (11) to implement the desired speed requirement, i.e., we define a CLF $V = (v - v_d)^2$, and choose $c_1 = c_2 = 1, c_3 = \epsilon > 0$ in Definition 5. Any control input should satisfy the CLF constraint (11).

We consider the HOCBF constraint (17) to implement the safety constraint (15), and consider the sufficient condition (48) introduced in the last section to guarantee the feasibility of the ACC problem. We use a HOCBF with $m = 1$ to impose this condition, as introduced in (37). We define $\alpha(\cdot)$ as a linear function in (37).

Finally, we use the discretization method introduced in the end of Section 2 to solve the ACC problem, i.e., we partition the time interval $[0, T]$ into a set of equal time intervals $\{[0, \Delta t), [\Delta t, 2\Delta t), \dots\}$, where $\Delta t > 0$. In each interval $[\omega\Delta t, (\omega+1)\Delta t)$ ($\omega = 0, 1, 2, \dots$), we assume the control is constant (i.e., the overall control will be piece-wise constant), and reformulate the ACC problem as a sequence of QPs. Specifically, at $t = \omega\Delta t$ ($\omega = 0, 1, 2, \dots$), we solve

$$\boldsymbol{u}^*(t) = \arg\min_{\boldsymbol{u}(t)} \frac{1}{2}\boldsymbol{u}(t)^T H\boldsymbol{u}(t) + F^T\boldsymbol{u}(t) \tag{49}$$

$$\boldsymbol{u}(t) = \begin{bmatrix} u(t) \\ \delta(t) \end{bmatrix}, H = \begin{bmatrix} \frac{2}{M^2} & 0 \\ 0 & 2p_{acc} \end{bmatrix}, F = \begin{bmatrix} \frac{-2F_r(v(t))}{M^2} \\ 0 \end{bmatrix}.$$

subject to

$$A_{\text{clf}}\boldsymbol{u}(t) \leq b_{\text{clf}},$$

$$A_{\text{limit}}\boldsymbol{u}(t) \leq b_{\text{limit}},$$

Table 1
Simulation parameters for the ACC problem.

Para.	Value	Units	Para.	Value	Units
$v(0)$	6	m/s	$z(0)$	100	m
v_p	13.89	m/s	v_d	24	m/s
M	1650	kg	g	9.81	m/s^2
f_0	0.1	N	f_1	5	Ns/m
f_2	0.25	Ns^2/m	l_0	10	m
Δt	0.1	s	ϵ	10	Unitless
$c_a(t)$	0.4	Unitless	$c_d(t)$	0.4	Unitless
p_{acc}	1	Unitless			

$$A_{\text{hocbf\_safety}}\boldsymbol{u}(t) \le b_{\text{hocbf\_safety}},$$

$$A_{\text{fea}}\boldsymbol{u}(t) \le b_{\text{fea}},$$

where $p_{acc} > 0$ and the constraint parameters are

$$A_{\text{clf}} = [L_g V(\boldsymbol{x}(t)), \quad -1],$$
$$b_{\text{clf}} = -L_f V(\boldsymbol{x}(t)) - \epsilon V(\boldsymbol{x}(t)).$$

$$A_{\text{limit}} = \begin{bmatrix} 1, & 0 \\ 1, & 0 \end{bmatrix},$$

$$b_{\text{limit}} = \begin{bmatrix} c_a Mg \\ -c_d Mg \end{bmatrix}.$$

$$A_{\text{hocbf\_safety}} = \left[\ \frac{1}{M}, \quad 0\ \right],$$

$$b_{\text{hocbf\_safety}} = \frac{F_r(v(t))}{M} + (p_1+p_2)(v_p - v(t)) + p_1 p_2 (z(t) - l_0)$$

$$A_{\text{fea}} = \left[\ \frac{p_1 p_2}{M(p_1+p_2)}, \quad 0\ \right],$$

$$b_{\text{fea}} = \frac{p_1 p_2 F_r(v(t))}{M(p_1+p_2)} + \frac{p_1 p_2}{p_1+p_2}(v_p - v(t)) + c_d g$$

After solving (49), we update (13) with $u^*(t)$, $\forall t \in (t_0 + \omega\Delta t, t_0 + (\omega+1)\Delta t)$.

The simulation parameters are listed in Table 1. We first present a case study in Fig. 4 showing that if the ego vehicle exceeds the speed constraint from the feasibility constraint (48), then the QP becomes infeasible. However, this infeasibility does not always hold since the feasibility constraint (48) is just a sufficient condition for the feasibility of QP (49). In order to show how the feasibility constraint (48) can be adapted to different parameters p_1, p_2 in (17), we vary them and compare the solution without this feasibility sufficient condition in the simulation, as shown in Figs. 5 and 6.

It follows from Figs. 5 and 6 that the QPs (49) are always feasible with the feasibility constraint (48) under different p_1, p_2, while the QPs may become infeasible without this constraint. This validates the effectiveness of the feasibility constraint. We also notice that the ego vehicle cannot reach the desired speed v_d with the feasibility condition (48); this is due to the fact that we are limiting the vehicle speed with (48). In order to make the ego vehicle reach the desired speed, we choose p_1, p_2 such that the following constraint is satisfied.

$$v_p + c_d g \frac{(p_1+p_2)}{p_1 p_2} \ge v_d \tag{50}$$

For example, the above constraint is satisfied when we select $p_1 = 0.5, p_2 = 1$ in this case. Then, the ego can reach the desired speed v_d, as the blue curves shown in Fig. 7.

We also compare the feasibility constraint (48) with the minimum braking distance approach from Ames et al. (2014). This

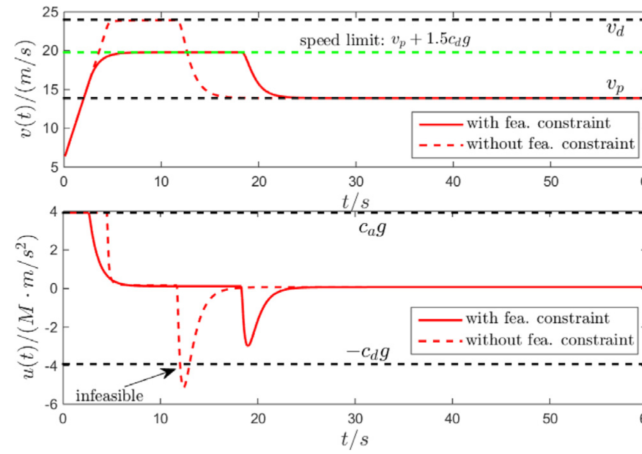

Fig. 4. A simple case with $p_1 = 1, p_2 = 2$. The QP becomes infeasible when the ego vehicle exceeds the speed limit $v_p + 1.5c_d g$ from (48).

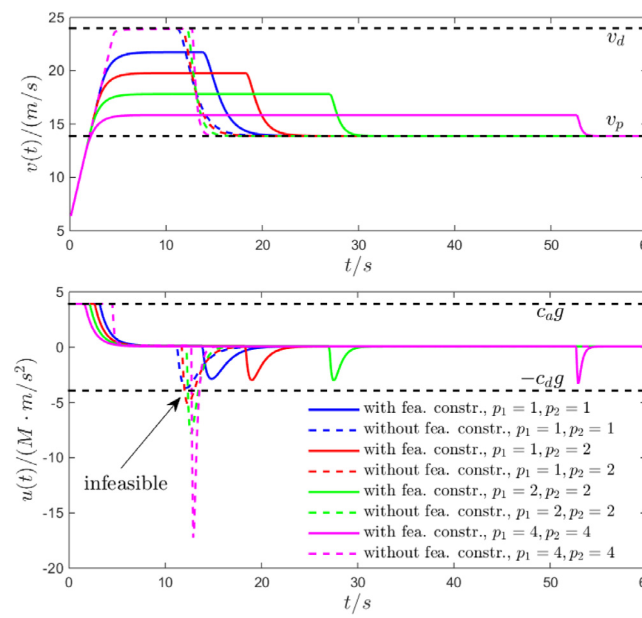

Fig. 5. Speed and control profiles for the ego vehicle under different p_1, p_2, with and without feasibility condition (48).

approach adds the minimum braking distance $\frac{0.5(v_p - v(t))^2}{c_d g}$ of the ego vehicle to the safety constraint (15):

$$z(t) \ge \frac{0.5(v_p - v(t))^2}{c_d g} + l_0, \forall t \ge 0. \tag{51}$$

Then, we can use a HOCBF with $m = 1$ (define $\alpha_1(\cdot)$ to be a linear function with slope 2 in Definition 4) to enforce the above constraint whose relative degree is one. As shown in Fig. 7, the HOCBF constraint for (51) conflicts with the control bounds, thus, the QP can still become infeasible. This is due to the fact that this approach adds an additional braking-distance-related constraint to the original problem, which could adversely decrease the problem feasibility as this new added constraint may conflict with existing control bounds. In contrast, our approach provides a novel way to make the new added feasibility constraint compliant with the existing constraints. This, therefore, can always guarantee feasibility once the sufficient conditions are determined.

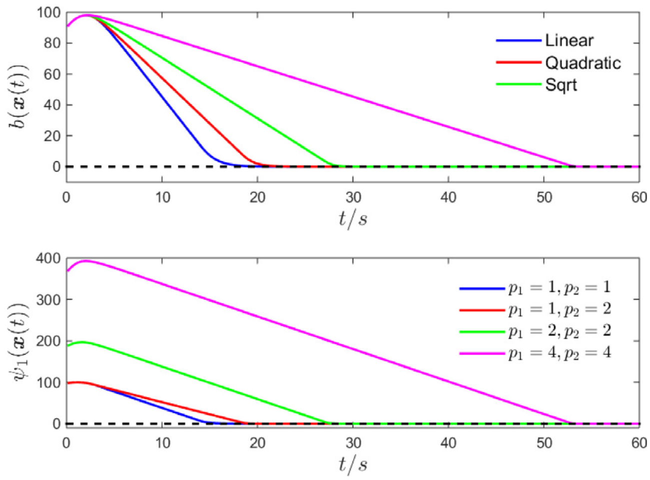

Fig. 6. The variation of functions $b(\boldsymbol{x}(t))$ and $\psi_1(\boldsymbol{x}(t))$ under different p_1, p_2. $b(\boldsymbol{x}(t)) \geq 0$ and $\psi_1(\boldsymbol{x}(t)) \geq 0$ imply the forward invariance of the set $C_1 \cap C_2$.

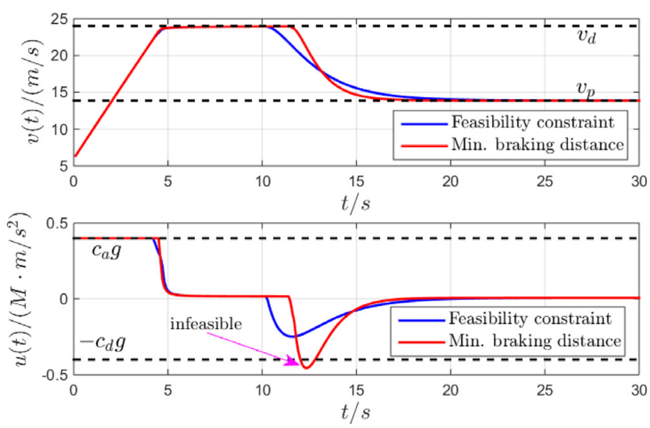

Fig. 7. Comparison between the feasibility constraint (48) with $p_1 = 0.5, p_2 = 1$ and the minimum braking distance approach from Ames et al. (2014). The HOCBF constraint for (51) in the minimum braking distance approach conflicts with the control bound (14).

6. Conclusion & future work

We provide provably correct sufficient conditions for feasibility guarantee of constrained optimal control problems in this paper. These conditions are found by the proposed feasibility constraint method. We have demonstrated the effectiveness of sufficient feasibility conditions by applying them to an adaptive cruise control problem. In the future, we will study the derivation of the necessary conditions of feasibility guarantee for constrained optimal control problems, or find less conservative sufficient conditions for specific dynamics. We will also try to figure out how to quickly find a single feasibility constraint for specific dynamics.

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