Euclidean metrics for motion generation on $SE(3)$

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Abstract: Previous approaches to trajectory generation for rigid bodies have been either based on the so-called invariant screw motions or on ad hoc decompositions into rotations and translations. This paper formulates the trajectory generation problem in the framework of Lie groups and Riemannian geometry. The goal is to determine optimal curves joining given points with appropriate boundary conditions on the Euclidean group. Since this results in a two-point boundary value problem that has to be solved iteratively, a computationally efficient, analytical method that generates near-optimal trajectories is derived. The method consists of two steps. The first step involves generating the optimal trajectory in an ambient space, while the second step is used to project this trajectory onto the Euclidean group. The paper describes the method, its applications and its performance in terms of optimality and efficiency.

Keywords: interpolation, lie groups, invariance, optimality

NOTATION

$\mathbf{A}$	homogeneous transformation matrix $\in SE(n)$
$\mathscr{A}$	acceleration vector
$\mathbf{B}$	affine transformation $\in GA(3)$
$\boldsymbol{d}$	position vector in $\{\boldsymbol{F}\}$
$\mathrm{D}/\mathrm{d}t$	covariant derivative
$\{\boldsymbol{F}\}$	reference (fixed) frame
$\mathbf{G}$	matrix of metric in $SO(3)$
$\tilde{\mathbf{G}}$	matrix of metric in $SE(3)$
$GA(n)$	affine group
$GL(n)$	general linear group of dimension n
$\boldsymbol{L}_i$	basis vector in $se(3)$
$\boldsymbol{L}_i^0$	basis vector in $so(3)$
$\mathbf{M}$	non-singular matrix $\in GL(3)$
$\{\boldsymbol{M}\}$	mobile frame
O	origin of $\{\boldsymbol{F}\}$
O'	origin of $\{\boldsymbol{M}\}$
$\mathbf{R}$	rotation matrix $\in SO(n)$
$\mathbb{R}^n$	Euclidean space of dimension n
$se(n)$	Lie algebra of $SE(n)$
$so(n)$	Lie algebra of $SO(n)$
S	twist $\in se(3)$
$SE(n)$	special Euclidean group
$SO(n)$	special orthogonal group
Tr	matrix trace
$T_P\mathbf{M}$	tangent space at $P \in \mathbf{M}$ to manifold $\mathbf{M}$
$\boldsymbol{v}$	linear velocity in $\{\boldsymbol{M}\}$
$\boldsymbol{V}$	velocity vector
x, y, z	Cartesian axes
$\boldsymbol{X}, \boldsymbol{Y}$	tangent vectors
σ	exponential coordinates on $SO(3)$
$\boldsymbol{\omega}$	angular velocity in $\{\boldsymbol{M}\}$
$\langle \cdot, \cdot \rangle$	Riemmanian metric

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1 INTRODUCTION

The problem of finding a smooth motion that interpolates between two given positions and orientations in $\mathbb{R}^3$ is well understood in Euclidean spaces [**1**, **2**], but it is not clear how these techniques can be generalized to curved spaces. There are two main issues that need to be addressed, particularly on non-Euclidean spaces. It is desirable that the computational scheme be independent of the description of the space and invariant with respect to the choice of the coordinate systems used to describe the motion. Secondly, the smoothness properties and the optimality of the trajectories need to be considered.

Shoemake [**3**] proposed a scheme for interpolating rotations with Bezier curves based on the spherical analogue of the de Casteljau algorithm. This idea was extended by Ge and Ravani [**4**] and Park and Ravani [**5**] to spatial motions. The focus in these articles is on the generalization of the notion of interpolation from the Euclidean space to a curved space.

Another class of methods is based on the representation of Bezier curves with Bernstein polynomials. Ge and Ravani [**6**] used the dual unit quaternion representation of $SE(3)$ and subsequently applied Euclidean methods to interpolate in this space. Jütler [**7**] formulated a more general version of the polynomial interpolation by using dual (instead of dual that) quaternions to represent $SE(6)$. In such a representation, an element of $SE(3)$ corresponds to a whole equivalence class of dual quaternions. Park and Kang [**8**] derived a rational interpolating scheme for the group of rotations $SO(3)$ by representing the group with Cayley parameters and using Euclidean methods in this parameter space. The advantage of these methods is that they produce rational curves.

It is worth noting that all these works (with the exception of reference [**5**]) use a particular coordinate representation of the group. In contrast, Noakes *et al.* [**9**] derived the necessary conditions for cubic splines on general manifolds without using a coordinate chart. These results are extended in reference [**10**] to the dynamic interpolation problem. Necessary conditions for higher-order splines are derived in reference [**11**]. A coordinate-free formulation of the variational approach was used to generate shortest paths and minimum acceleration and jerk trajectories on $SO(3)$ and $SE(3)$ in reference [**12**]. However, analytical solutions are available only in the simplest of cases, and the procedure for solving optimal motions, in general, is computationally intensive. If optimality is sacrificed, it is possible to generate bi-invariant trajectories for interpolation and approximation using the exponential map on the Lie algebra [**13**]. While the solutions are of closed form, the resulting trajectories have no optimality properties.

This paper is built on the results from references [**12**] and [**13**]. It is shown that a left or right invariant metric on $SO(3)[SE(3)]$ is inherited from the higher-dimensional manifold $GL(3)[GA(3)]$ equipped with the appropriate metric. Next, a projection operator is defined and subsequently used to project optimal curves from the ambient manifold onto $SO(3)[SE(3)]$. It is proved that the geodesic on $SO(3)$ and the projected geodesic from $GL(3)$ follow the same path, but with a different parameterization. The line from $GL(3)$ is then shown to be parameterizable to yield the exact geodesic on $SO(3)$ by projection. Several examples are presented to illustrate the merits of the method and to show that it produces near-optimal results, especially when the excursion of the trajectories is 'small'.

2 BACKGROUND

2.1 Lie groups $SO(3)$ and $SE(3)$

Let $GL(n)$ denote the general linear group of dimension n. As a manifold, $GL(n)$ can be regarded as an open subset of $\mathbb{R}^{n^2}$. Moreover, matrix multiplication and inversion are both smooth operations, which make $GL(n)$ a Lie group. The special orthogonal group is a subgroup of the general linear group, defined as

$$SO(n) = \{\mathbf{R} | \mathbf{R} \in GL(n), \mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I}, \det\mathbf{R} = 1\}$$

where $SO(n)$ is referred to as the rotation group on $\mathbb{R}^n$, $GA(n) = GL(n) \times \mathbb{R}^n$ is the affine group and $SE \times (n) = SO(n) \times \mathbb{R}^n$ is the special Euclidean group and is the set of all rigid displacements in $\mathbb{R}^n$. Special consideration will be given to $SO(3)$ and $SE(3)$.

Consider a rigid body moving in free space. Assume an inertial reference frame $\{\boldsymbol{F}\}$ fixed in space and a frame $\{\boldsymbol{M}\}$ fixed to the body at point O' as shown in Fig. 1. At each instance, the configuration (position and orientation) of the rigid body can be described by a homogeneous transformation matrix, $\mathbf{A}$, corresponding to the displacement from frame $\{\boldsymbol{F}\}$ to frame $\{\boldsymbol{M}\}$. $SE(3)$ is the set of all rigid body transformations in three dimensions:

$$SE(3) = \left\{\mathbf{A} | \mathbf{A} = \begin{bmatrix} \mathbf{R} & \boldsymbol{d} \\ 0 & 1 \end{bmatrix}, \mathbf{R} \in SO(3), \boldsymbol{d} \in \mathbb{R}^3\right\}$$

$SE(3)$ is a closed subset of $GA(3)$, and therefore a Lie group.

On any Lie group the tangent space at the group identity has the structure of a Lie algebra. The Lie algebras of $SO(3)$ and $SE(3)$, denoted by $so(3)$ and $se(3)$ respectively, are given by

$$so(3) = \left\{\hat{\boldsymbol{\omega}} | \hat{\boldsymbol{\omega}} \in \mathbb{R}^{3\times 3}, \hat{\boldsymbol{\omega}}^{\mathrm{T}} = -\hat{\boldsymbol{\omega}}\right\}$$

$$se(3) = \left\{\begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ 0 & 0 \end{bmatrix} | \hat{\boldsymbol{\omega}} \in \mathbb{R}^{3\times 3}, \boldsymbol{v} \in \mathbb{R}^3, \hat{\boldsymbol{\omega}}^{\mathrm{T}} = -\hat{\boldsymbol{\omega}}\right\}$$

A 3×3 skew-symmetric matrix $\hat{\boldsymbol{\omega}}$ can be uniquely identified with a vector $\boldsymbol{\omega} \in \mathbb{R}^3$ so that for an arbitrary

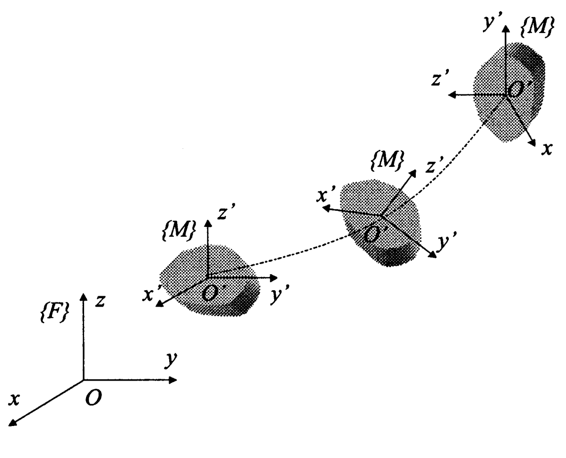

Fig. 1 Inertial (fixed) frame and the moving frame attached to the rigid body

vector $\boldsymbol{x} \in \mathbb{R}^3$, $\hat{\boldsymbol{\omega}}\boldsymbol{x} = \boldsymbol{\omega} \times \boldsymbol{x}$, where $\times$ is the vector cross-product operation in $\mathbb{R}^3$. Each element $S \in se(3)$ can thus be identified with a vector pair $\{\boldsymbol{\omega}, \boldsymbol{v}\}$. Given a curve

$$\mathbf{A}(t) : [-a, a] \to SE(3), \ \mathbf{A}(t) = \begin{bmatrix} \mathbf{R}(t) & \boldsymbol{d}(t) \\ 0 & 1 \end{bmatrix}$$

an element $S(t)$ of the Lie algebra $se(3)$ can be associated with the tangent vector $\dot{\boldsymbol{A}}(t)$ at an arbitrary point t by

$$S(t) = \mathbf{A}^{-1}(t)\dot{\boldsymbol{A}}(t) = \begin{bmatrix} \hat{\boldsymbol{\omega}}(t) & \mathbf{R}^{\mathrm{T}}\dot{\boldsymbol{d}} \\ 0 & 0 \end{bmatrix} \tag{1}$$

where $\hat{\boldsymbol{\omega}}(t) = \mathbf{R}^{\mathrm{T}}\dot{\mathbf{R}}$ is the corresponding element from $so(3)$.

A curve on $SE(3)$ physically represents a motion of the rigid body. If $\{\boldsymbol{\omega}(t), \boldsymbol{v}(t)\}$ is the vector pair corresponding to $S(t)$, then $\boldsymbol{\omega}$ physically corresponds to the angular velocity of the rigid body while $\boldsymbol{v}$ is the linear velocity of the origin O' of the frame $\{\boldsymbol{M}\}$, both expressed in the frame $\{\boldsymbol{M}\}$. In kinematics, elements of this form are called twists and $se(3)$ thus corresponds to the space of twists. The twist $S(t)$ computed from equation (1) does not depend on the choice of the inertial frame $\{\boldsymbol{F}\}$. For this reason, $S(t)$ is called the left invariant representation of the tangent vector $\dot{\boldsymbol{A}}$.

The standard basis for the vector space $so(3)$ is

$$\boldsymbol{L}_1^0 = \hat{\boldsymbol{e}}_1, \qquad \boldsymbol{L}_2^0 = \hat{\boldsymbol{e}}_2, \qquad \boldsymbol{L}_3^0 = \hat{\boldsymbol{e}}_3 \tag{2}$$

where

$$\boldsymbol{e}_1 = [1\ 0\ 0]^{\mathrm{T}}, \qquad \boldsymbol{e}_2 = [0\ 1\ 0]^{\mathrm{T}}, \qquad \boldsymbol{e}_3 = [0\ 0\ 1]^{\mathrm{T}}$$

and $\boldsymbol{L}_1^0, \boldsymbol{L}_2^0$ and $\boldsymbol{L}_3^0$ represent instantaneous rotations about the Cartesian axes x, y and z respectively. The components of a $\hat{\boldsymbol{\omega}} \in se(3)$ in this basis are given precisely by the angular velocity vector $\boldsymbol{\omega}$.

The standard basis for $se(3)$ is

$$\boldsymbol{L}_1 = \begin{bmatrix} \boldsymbol{L}_1^0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{L}_2 = \begin{bmatrix} \boldsymbol{L}_2^0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{L}_3 = \begin{bmatrix} \boldsymbol{L}_3^0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\boldsymbol{L}_4 = \begin{bmatrix} 0 & \boldsymbol{e}_1 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{L}_5 = \begin{bmatrix} 0 & \boldsymbol{e}_2 \\ 0 & 0 \end{bmatrix}, \qquad \boldsymbol{L}_6 = \begin{bmatrix} 0 & \boldsymbol{e}_3 \\ 0 & 0 \end{bmatrix}$$

The twists $\boldsymbol{L}_4, \boldsymbol{L}_5$ and $\boldsymbol{L}_6$ represent instantaneous translations along the Cartesian axes x, y and z respectively. The components of a twist $S \in se(3)$ in this basis are given precisely by the velocity vector pair $\{\omega, \boldsymbol{v}\}$.

2.2 Left invariant vector fields

A *differentiable vector field* is a smooth assignment of a tangent vector to each element of the manifold. An example of a differentiable vector field, $\boldsymbol{X}$, on $SE(3)$ is obtained by left translation of an element $S \in se(3)$. The value of the vector field $\boldsymbol{X}$ at an arbitrary point $\mathbf{A} \in SE(3)$ is given by

$$\boldsymbol{X}(\mathbf{A}) = \bar{S}(\mathbf{A}) = \mathbf{A}S \tag{4}$$

A vector field generated by equation (4) is called a left invariant vector field and the notation $\bar{S}$ is used to indicate that the vector field was obtained by left translating the Lie algebra element S.

Since the vectors $\boldsymbol{L}_1, \boldsymbol{L}_2, \ldots, \boldsymbol{L}_6$ are a basis for the Lie algebra $se(3)$, the vectors $\boldsymbol{L}_1(A), \ldots, \boldsymbol{L}_6(A)$ form a basis of the tangent space at any point $\mathbf{A} \in SE(3)$. Therefore, any vector field $\boldsymbol{X}$ can be expressed as

$$\boldsymbol{X} = \sum_{i=1}^{6} X^i \boldsymbol{L}_i \tag{5}$$

where the coefficients X^i vary over the manifold. If the coefficients are constants, then $\boldsymbol{X}$ is left invariant. By defining

$$\boldsymbol{\omega} = [X^1, X^2, X^3]^{\mathrm{T}}, \qquad \boldsymbol{v} = [X^4, X^5, X^6]^{\mathrm{T}}$$

a vector pair of functions $\{\boldsymbol{\omega}, \boldsymbol{v}\}$ can be associated with an arbitrary vector field $\boldsymbol{X}$. If a curve $\mathbf{A}(t)$ describes a motion of the rigid body and $\boldsymbol{V} = \mathrm{d}\mathbf{A}/\mathrm{d}t$ is the vector field tangent to $\mathbf{A}(t)$, the vector pair $\{\boldsymbol{\omega}, \boldsymbol{v}\}$ associated with $\boldsymbol{V}$ corresponds to the instantaneous twist (screw axis) for the motion. In general, the twist $\{\boldsymbol{\omega}, \boldsymbol{v}\}$ changes with time.

2.3 Riemannian metrics on Lie groups

If a smoothly varying, positive definite, bilinear, symmetric form $\langle \cdot, \cdot \rangle$ is defined on the tangent space at each point on the manifold, such a form is called a Riemannian metric and the manifold is Riemannian [**14**]. On an n-dimensional manifold, the metric is locally characterized by an $n \times n$ matrix of C^∞ functions $\mathbf{g}_{ij} = \langle X_i, X_j \rangle$, where X_i are basis vector fields. If the basis vector fields can be defined globally, then the matrix $[\mathbf{g}_{ij}]$ completely defines the metric.

On $SE(3)$ (on any Lie group), an inner product on the Lie algebra can be extended to a Riemannian metric over the manifold using left (or right) translation. To see this, consider the inner product of two elements S_1, $S_2 \in se(3)$ defined by

$$\langle S_1, S_2 \rangle|_{\mathbf{I}} = \boldsymbol{s}_1^{\mathrm{T}} \mathbf{G} \boldsymbol{s}_2 \tag{6}$$

where $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are the 6×1 vectors of components of S_1 and S_2 with respect to some basis and $\mathbf{G}$ is a positive definite matrix. If $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are tangent vectors at an arbitrary group element $\mathbf{A} \in SE(3)$, the inner product $\langle \boldsymbol{V}_1, \boldsymbol{V}_2 \rangle|_{\mathbf{A}}$ in the tangent space $T_{\mathbf{A}}SE(3)$ can be defined by

$$\langle \boldsymbol{V}_1, \boldsymbol{V}_2 \rangle|_{\mathbf{A}} = \langle \mathbf{A}^{-1}\boldsymbol{V}_1, \mathbf{A}^{-1}\boldsymbol{V}_2 \rangle|_{\mathbf{I}} \tag{7}$$

The metric obtained in such a way is said to be left invariant [**14**].

2.4 Affine connection, covariant derivative and geodesic flow

Any motion of a rigid body is described by a smooth curve $\mathbf{A}(t) \in SE(3)$. The velocity is the tangent vector to the curve $\boldsymbol{V}(t) = \mathrm{d}A/\mathrm{d}t(t)$.

An *affine connection* on $SE(3)$ is a map that assigns to each pair of C^∞ vector fields $\boldsymbol{X}$ and $\boldsymbol{Y}$ on $SE(3)$ another C^∞ vector field $\nabla_{\boldsymbol{X}}\boldsymbol{Y}$ which is $\mathbb{R}$-bilinear in $\boldsymbol{X}$ and $\boldsymbol{Y}$ and, for any smooth real function f on $SE(3)$, satisfies $\nabla_{f\boldsymbol{X}}\boldsymbol{Y} = f\nabla_{\boldsymbol{X}}\boldsymbol{Y}$ and $\nabla_{\boldsymbol{X}} f\boldsymbol{Y} = f\nabla_{\boldsymbol{X}}\boldsymbol{Y} + \boldsymbol{X}(f)\boldsymbol{Y}$.

The *Christoffel symbols* $\boldsymbol{\Gamma}^i_{jk}$ of the connection at a point $\boldsymbol{A} \in SE(3)$ are defined by $\nabla_{\boldsymbol{L}_j}\boldsymbol{L}_k = \boldsymbol{\Gamma}^i_{jk}\boldsymbol{L}_i$, where $\boldsymbol{L}_1, \ldots, \boldsymbol{L}_6$ is the basis in $T_{\mathbf{A}}SE(3)$ and the summation is understood.

If $\boldsymbol{A}(t)$ is a curve and $\boldsymbol{X}$ is a vector field, the *covariant derivative* of $\boldsymbol{X}$ along $\mathbf{A}$ is defined by

$$\frac{\mathrm{D}\boldsymbol{X}}{\mathrm{d}t} = \nabla_{\dot{\boldsymbol{A}}(t)\boldsymbol{X}}$$

$\boldsymbol{X}$ is said to be *autoparallel* along $\mathbf{A}$ if $\mathrm{D}\boldsymbol{X}/\mathrm{d}t = 0$. A curve $\mathbf{A}$ is a *geodesic* if $\dot{\boldsymbol{A}}$ is autoparallel along $\mathbf{A}$. An equivalent characterization of a geodesic is the following set of equations:

$$\ddot{a}^i + \Gamma^i_{jk}\dot{a}^j\dot{a}^k = 0 \tag{8}$$

where $a_i, i = 1, \ldots, 6$ is an arbitrary set of local coordinates on $SE(3)$.

For a manifold with a Riemannian (or pseudo-Riemannian) metric, there exists a unique symmetric connection which is compatible with the metric [**14**]. Given a connection, the acceleration and higher derivatives of the velocity can be defined. The acceleration, $\mathscr{A}(t)$, is the covariant derivative of the velocity along the curve:

$$\mathscr{A} = \frac{\mathrm{D}}{\mathrm{d}t}\left(\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right) = \nabla_{\boldsymbol{V}}\boldsymbol{V} \tag{9}$$

2.5 Exponential map and local parameterization of $SE(3)$

If $\mathbf{M}$ is a manifold with a connection ∇, the *exponential map* at an arbitrary $q \in \mathbf{M}$ is defined as follows. Let $\gamma_{\boldsymbol{V}}(t)$ be the unique geodesic passing through q at $t = 0$ with velocity $\boldsymbol{V}$, i.e. $\gamma_{\boldsymbol{V}}(0) = q$ and $\dot{\gamma}_{\boldsymbol{V}}(0) = \boldsymbol{V}$. Then, by definition, $\exp_q$ maps $\boldsymbol{V} \in T_q\mathbf{M}$ to the point $\gamma_{\boldsymbol{V}}(1) \in \mathbf{M}$. Using homogeneity of geodesics, it is easy to prove [**14**] that $\gamma_{t\boldsymbol{V}}(s) = \gamma_{\boldsymbol{V}}(ts)$ which gives $\exp_q(t\boldsymbol{V}) = \gamma_{\boldsymbol{V}}(t)$. Also, $\exp_q$ is a diffeomorphism of a neighbourhood of $0 \in T_q\mathbf{M}$ to a neighborhood of $q \in \mathbf{M}$. This gives a local chart for $\mathbf{M}$ called *normal coordinates*. These coordinates are convenient for computations (as in this work) because rays through 0 are geodesics.

The exponential map on $SO(3)$ with metric $\mathbf{G} = \alpha\mathbf{I}$ is given special consideration in this paper. For $\mathbf{R} \in SO(3)$ and $\boldsymbol{V} \in T_{\mathbf{R}}SO(3)$, it is possible to define $\exp_{\mathbf{R}}(\boldsymbol{V}) = Re^{\mathbf{R}^{\mathrm{T}}}\boldsymbol{V}$. If $\boldsymbol{v} = [v_1 v_2 v_3]$ is the expansion of $\boldsymbol{V}$ in the local basis of $T_{\mathbf{R}}SO(3)$ (i.e. $\boldsymbol{V} = v_1\boldsymbol{L}^0_1 + v_2\boldsymbol{L}^0_2 + v_3\boldsymbol{L}^0_3$), it is easy to see that $\exp_{\mathbf{R}}(\boldsymbol{V}) = Re^{\hat{v}}$. As a special case, for $S \in so(3)$, $\exp_{\mathbf{I}}(S) = \boldsymbol{e}^{\hat{\sigma}}$, where $\sigma = [\sigma_1\sigma_2\sigma_3]$ is the expansion of S in the basis $\boldsymbol{L}^0_1, \boldsymbol{L}^0_2, \boldsymbol{L}^0_3$. This gives a local parameterization of $SO(3)$ around identity known as *exponential coordinates*.

In this paper, a parameterization of $SE(3)$ induced by the product structure $SO(3) \times \mathbb{R}^3$ is chosen. In other words, a set of coordinates σ_1, σ_2, σ_3, d_1, d_2, d_3 for an arbitrary element $\mathbf{A} = (\mathbf{R}, \boldsymbol{d}) \in SE(3)$ is defined so that d_1, d_2, d_3 are the coordinates of $\boldsymbol{d}$ in $\mathbb{R}^3$. Exponential coordinates, as defined in Section 2.5, are chosen as the local parameterization of $SO(3)$. For $\mathbf{R} \in SO(3)$ sufficiently close to the identity [i.e excluding the points $\mathrm{Tr}(\mathbf{R}) = -1(\mathrm{Tr}(\mathbf{A}) = 0$, or, equivalently, rotations through angles up to π], the exponential coordinates are given by

$$\mathbf{R} = \boldsymbol{e}^{\hat{\sigma}}, \sigma \in \mathbb{R}^3$$

2.6 Screw motions

One of the fundamental results in rigid body kinematics was proved by Chasles at the beginning of the nineteenth century: 'Any rigid body displacement can be realized by a rotation about an axis combined with a translation parallel to that axis.' Note that a displacement must be understood as an element of $SE(3)$, while a motion is a curve on $SE(3)$. If the rotation from Chasles's theorem is performed at constant angular velocity and the translation at constant translational velocity, the motion leading to the displacement becomes a *screw motion*. Chasles's theorem then says that 'any rigid body displacement can be realized by a screw motion'.

A curve $\mathbf{A}(t)$ on a Lie group is called a *one-parameter subgroup* if $\mathbf{A}(t+s) = \mathbf{A}(t)\mathbf{A}(s)$. The following are equivalent ways of defining a screw motion $\mathbf{A}(t) \in SE(3)$:

1. $\mathbf{A}^{-1}(t)\dot{\boldsymbol{A}}(t)$ is constant.
2. $\{\boldsymbol{\omega}, \boldsymbol{v}\}$ is constant.
3. $\mathbf{A}(t)$ is a one-parameter subgroup of $SE(3)$.
4. The tangent vectors $\dot{\boldsymbol{A}}(t)$ to the curve form a left invariant vector field.

With this mathematical definition, Chasles's theorem can be restated in the form: 'For every element in $SE(3)$ different from identity, there is a unique one-parameter subgroup to which that element belongs.' Note that the definition of a one-parameter subgroup is not dependent on a metric.

Given two end positions on $SE(3)$, it can be concluded that there always exists an interpolating screw motion. Is this motion physically meaningful and/or optimal from some point of view? To talk about optimality, a metric on the manifold must first be found. Optimal interpolating motions with respect to a given metric are geodesics, minimum acceleration curves, minimum jerk curves and so on.

What is the connection between geodesics as defined in Section 2.4 and screw motions (one-parameter subgroups)? The following result is true for any Lie group [**14**]: for a bi-invariant metric, the geodesics that start from identity are one-parameter subgroups.

As a particular case, geodesics through identity on $SO(3)$ with metric $\mathbf{G} = \alpha \boldsymbol{I}$ are one-parameter subgroups (ω = constant). Also, for the bi-invariant semi-Riemannian metric on $SE(3)$

$$\mathbf{G} = \begin{bmatrix} \alpha\mathbf{I}_3 & \beta\mathbf{I}_3 \\ \beta\mathbf{I}_3 & 0 \end{bmatrix}, \qquad \alpha, \beta > 0 \tag{10}$$

geodesics through identity are screw motions. The conclusion is that an interpolating screw motion is not the appropriate choice if the metric on $SE(3)$ is different from the bi-invariant metric (10), which is the case of the kinetic energy metric.

3 RIEMANNIAN METRICS ON $SO(3)$ AND $SE(3)$

In this section it is shown that there is a simple way of defining a left or right invariant metric in $SO(3)$ [$SE(3)$] by introducing an appropriate constant metric in $GL(3)$ [$GA(3)$]. Defining a metric at the Lie algebra $so(3)$ [or $se(3)$] and extending it through left (right) translations is equivalent to inheriting the appropriate metric from the ambient manifold at each point. In this paper, only metrics on $SE(3)$ that are products of the bi-invariant metric on $SO(3)$ and the Euclidean metric on $\mathbb{R}^3$ are considered. A more general treatment accommodating arbitrary metrics on $SO(3)$ is to be published elsewhere.

3.1 Metrics on $GL(3)$ and $SO(3)$

For any $\mathbf{M} \in GL(3)$ and any $\boldsymbol{X}, \boldsymbol{Y} \in T_{\mathbf{M}}GL(3)$, define

$$\langle \boldsymbol{X}, \boldsymbol{Y} \rangle_{GL} = \mathrm{Tr}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{Y}) \tag{11}$$

where Tr denotes the trace of a square matrix and $T_M GL(3)$ is the tangent space to $GL(3)$ at $\mathbf{M}$, which is isomorphic to $GL(3)$. By definition, form (11) is the same at all points in $GL(3)$. It is easy to see that $\langle , \rangle_{GL}$ is a positive definite quadratic form in the entries of $\boldsymbol{X}$ and $\boldsymbol{Y}$, and therefore a metric. This induces the Euclidean norm on $T_{\mathbf{M}}GL(3)$, which is also called the Frobenius matrix norm.

Proposition. The metric given by (11) defined on $GL(3)$ is bi-invariant when restricted to $SO(3)$.

Proof. Let any $\mathbf{M} \in GL(3)$ and any vectors $\boldsymbol{X}$, $\boldsymbol{Y}$ in the tangent space at an arbitrary point of $GL(3)$. Then

$$\langle \boldsymbol{X}, \boldsymbol{Y} \rangle_{GL} = \mathrm{Tr}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{Y}),$$
$$\langle \mathbf{M}\boldsymbol{X}, \mathbf{M}\boldsymbol{Y} \rangle_{GL} = \mathrm{Tr}(\boldsymbol{X}^{\mathrm{T}}\mathbf{M}^{\mathrm{T}}\mathbf{M}\boldsymbol{Y})$$

from it can be concluded that the metric is invariant under left translations with elements from $SO(3)$. Therefore, when restricted to $SO(3)$, the metric becomes left invariant. For right invariance, if $\mathbf{R} \in SO(3)$, then

$$\langle \boldsymbol{X}, \boldsymbol{Y} \rangle_{GL} = \mathrm{Tr}(\boldsymbol{Y}\boldsymbol{X}^{\mathrm{T}}),$$
$$\langle \boldsymbol{X}\mathbf{R}, \boldsymbol{Y}\mathbf{R} \rangle_{GL} = \mathrm{Tr}(\boldsymbol{Y}\mathbf{R}\mathbf{R}^{\mathrm{T}}\boldsymbol{X}^{\mathrm{T}}) = \mathrm{Tr}(\boldsymbol{Y}\boldsymbol{X}^{\mathrm{T}})$$

and the claim is proved. End of proof.

To find the induced metric on $SO(3)$, let $\mathbf{R}$ be an arbitrary element from $SO(3)$, $\boldsymbol{X}$, $\boldsymbol{Y}$ be two vectors from $T_{\mathbf{R}}SO(3)$ and $\boldsymbol{R}_x(t), \boldsymbol{R}_y(t)$ be the corresponding local flows so that

$$\boldsymbol{X} = \dot{\boldsymbol{R}}_x(0), \qquad \boldsymbol{Y} = \dot{\boldsymbol{R}}_y(0), \qquad \boldsymbol{R}_x(0) = \boldsymbol{R}_y(0) = \boldsymbol{R}$$

The metric inherited from $GL(3)$ can be written as

$$\langle \boldsymbol{X}, \boldsymbol{Y} \rangle_{SO} = \langle \boldsymbol{X}, \boldsymbol{Y} \rangle_{GL} = \mathrm{Tr}(\dot{\boldsymbol{R}}_x^{\mathrm{T}}(0)\dot{\boldsymbol{R}}_y(0))$$
$$= \mathrm{Tr}(\dot{\boldsymbol{R}}_x^{\mathrm{T}}(0)\mathbf{R}\mathbf{R}^{\mathrm{T}}\dot{\boldsymbol{R}}_y(0)) = \mathrm{Tr}(\hat{\omega}_x^{\mathrm{T}}\hat{\boldsymbol{\omega}}_y)$$

where $\hat{\boldsymbol{\omega}}_x = \boldsymbol{R}_x(0)^{\mathrm{T}}\dot{\boldsymbol{R}}_x(0)$ and $\hat{\boldsymbol{\omega}}_y = \boldsymbol{R}_y(0)^{\mathrm{T}}\dot{\boldsymbol{R}}_y(0)$ are the corresponding twists from the Lie algebra $so(3)$. If the above relation is written using the vector form of the twists, some elementary algebra leads to

$$\langle \boldsymbol{X}, \boldsymbol{Y} \rangle_{so} = 2\boldsymbol{\omega}_x^{\mathrm{T}}\boldsymbol{\omega}_y \tag{12}$$

A different equivalent way of arriving at expression (12) would be defining the metric in $so(3)$ [i.e. at identity of $SO(3)$] as being the one inherited from $T_I GL(3)$:

$$\mathbf{g}_{ij} = \mathrm{Tr}(\boldsymbol{L}_i^{0\,T}\boldsymbol{L}_j^0) = \delta'_{ij}, \qquad i,j = 1,2,3$$

where $\boldsymbol{L}_1^0, \boldsymbol{L}_2^0, \boldsymbol{L}_3^0$ is the basis in $so(3)$ and δ_{ij} is the Kronecker symbol. Left or right translating this metric throughout the manifold is equivalent to inheriting the metric at each three-dimensional tangent space of $SO(3)$ from the corresponding nine-dimensional tangent space of $GL(3)$.

Remark 1. The matrix $\boldsymbol{G}$ of the metric as defined in (6) is $\mathbf{G} = 2\mathbf{I}$, which is the standard scale-independent bi-invariant metric on $SO(3)$. This is consistent with the above proposition.

Remark 2. The metric given by (12) can be interpreted as the (rotational) kinetic energy metric of a spherical rigid body.

3.2 Metrics on $GA(3)$ and $SE(3)$

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two vectors from the tangent space at an arbitrary point of $GA(3)$ ($\boldsymbol{X}$ and $\boldsymbol{Y}$ are 4×4 matrices with all entries of the last row equal to zero). Similarly to Section 3.1, a quadratic form defined by

$$\langle \boldsymbol{X}, \boldsymbol{Y}\rangle_{GA} = \mathrm{Tr}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{Y}) \tag{13}$$

is a point-independent Riemmanian metric on $GA(3)$.

It is possible to obtain a left invariant metric on $SE(3)$ by inheriting the metric $\langle\,\cdot\,\rangle_{GA}$ given by (13) from $GA(3)$. To derive the induced metric in $SE(3)$, the same procedure as in Section 3.1 is followed.

Let $\mathbf{A}$ be an arbitrary element from $SE(3)$. Let $\boldsymbol{X}$, $\boldsymbol{Y}$ be two vectors from $T_{\mathbf{A}}SE(3)$ and $\boldsymbol{A}_x(t)$ and $\boldsymbol{A}_y(t)$ the corresponding local flows so that

$$\boldsymbol{X} = \dot{\boldsymbol{A}}_x(0), \qquad \boldsymbol{Y} = \dot{\boldsymbol{A}}_y(0), \qquad \boldsymbol{A}_x(0) = \boldsymbol{A}_y(0) = \mathbf{A}$$

Let

$$\boldsymbol{A}_i(t) = \begin{bmatrix} \boldsymbol{R}_i(t) & \boldsymbol{d}_i(t) \\ 0 & 1 \end{bmatrix}, \qquad i \in \{x, y\}$$

and the corresponding twists at time 0

$$S_i = \boldsymbol{A}_i^{-1}(0)\dot{\boldsymbol{A}}_i(0) = \begin{bmatrix} \hat{\boldsymbol{\omega}}_i & \boldsymbol{v}_i \\ 0 & 0 \end{bmatrix}, \qquad i \in \{x, y\}$$

The metric inherited from $GA(3)$ can be written as

$$\begin{aligned}\langle \boldsymbol{X},\ \boldsymbol{Y}\rangle_{SE} = \langle \boldsymbol{X}, \boldsymbol{Y}\rangle_{GA} &= \mathrm{Tr}(\dot{\boldsymbol{A}}_x^{\mathrm{T}}(0)\dot{\boldsymbol{A}}_y(0)) \\ &= \mathrm{Tr}(S_x^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}S_y)\end{aligned}$$

Now, using the orthogonality of the rotational part of $\mathbf{A}$ and the special form of the twist matrices, straightforward calculations lead to

$$\langle \boldsymbol{X}, \boldsymbol{Y}\rangle_{SE} = \mathrm{Tr}(S_x^T S_y) = \mathrm{Tr}(\hat{\boldsymbol{\omega}}_x^{\mathrm{T}}\hat{\boldsymbol{\omega}}_y) + \boldsymbol{v}_x^{\mathrm{T}}\boldsymbol{v}_y$$

or, equivalently,

$$\langle \boldsymbol{X}, \boldsymbol{Y}\rangle_{SE} = [\boldsymbol{\omega}_x^{\mathrm{T}}\boldsymbol{v}_x^{\mathrm{T}}]\bar{\mathbf{G}}\begin{bmatrix}\boldsymbol{\omega}_y \\ \mathbf{v}_y\end{bmatrix}, \qquad \bar{\mathbf{G}} = \begin{bmatrix} 2I_3 & 0 \\ 0 & \mathbf{I}_3\end{bmatrix} \tag{14}$$

Remark 1. Metric (14) is the scale-independent metric on $SE(3)$ proposed by Park and Brockett [**15**] for $\alpha = 2$ and $\beta = 1$. It is a product metric and has been extensively studied in reference [**12**].

Remark 2. Straightforward calculations show that $SE(3)$ can be provided with the same metric (14) by inheriting the metric from the ambient space at $se(3)$:

$$\mathbf{g}_{ij} = \mathrm{Tr}(\boldsymbol{L}_i^{\mathrm{T}}\boldsymbol{L}_j) = \mathbf{g}_{ij} = \begin{cases} 2\delta_{ij}, & i,j = 1,2,3 \\ \delta_{ij}, & i,j = 4,5,6 \\ 0, & \text{elsewhere}\end{cases}$$

and left translating it throughout the manifold. Therefore, the metric $\mathrm{Tr}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{Y})$ from $GA(3)$ becomes left invariant when restricted to $SE(3)$.

Remark 3. Metric (14) can be interpreted as being the kinetic energy of a moving (rotating and translating) spherical rigid body when the body fixed frame $\{\boldsymbol{M}\}$ is placed at the centroid of the body and aligned with its principal axes.

4 PROJECTION ON $SO(3)$

The norm induced by metric (11) can be used to define the distance between elements in $GL(3)$. Using this distance, for a given $\boldsymbol{M} \in GL(3)$, the *projection* of $\mathbf{M}$ on $SO(3)$ is defined as being the closest $\mathbf{R} \in SO(3)$ with respect to norm $\|\cdot\|_{GL}$. The following proposition gives the solution of the projection problem for the general case of $GL(n)$:

Proposition 1. Let $\mathbf{M} \in GL(n)$ and $\mathbf{M} = \mathbf{U}\Sigma\boldsymbol{V}^{\mathrm{T}}$ be its singular value decomposition. Then the projection of $\mathbf{M}$ on $SO(n)$ is given by $\mathbf{R} = U\boldsymbol{V}^{\mathrm{T}}$.

Proof. The problem to be solved is a minimization problem:

$$\min_{\mathbf{R}\in SO(n)} \|\mathbf{M} - \mathbf{R}\|_{GL}^2$$

If $\mathbf{M} = \mathbf{U}\Sigma\boldsymbol{V}^{\mathrm{T}}$, then

$$\begin{aligned}\|\mathbf{M} - \mathbf{R}\|_{GL}^2 &= \mathrm{Tr}[(\mathbf{M} - \mathbf{R})^{\mathrm{T}}(\mathbf{M} - \mathbf{R})] \\ &= \mathrm{Tr}(\mathbf{M}^{\mathrm{T}}\mathbf{M} - \mathbf{M}^{\mathrm{T}}\mathbf{R} - \mathbf{R}^{\mathrm{T}}\mathbf{M} + \mathbf{R}^{\mathrm{T}}\mathbf{R})\end{aligned}$$

Note that $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$, $\mathrm{Tr}(\mathbf{M}^{\mathrm{T}}\mathbf{R}) = \mathrm{Tr}(\mathbf{R}^{\mathrm{T}}\mathbf{M})$ and the quantity $\mathbf{M}^{\mathrm{T}}\mathbf{M}$ is a constant and therefore does not affect the optimization. Therefore, the problem to be solved becomes

$$\max_{\mathbf{R}\in SO(n)} \mathrm{Tr}(\mathbf{M}^{\mathrm{T}}\mathbf{R})$$

Let $\Sigma = \mathrm{diag}\{\sigma_1, \ldots, \sigma_n\}$ and $\boldsymbol{C} = \mathbf{R}^{\mathrm{T}}\mathbf{U}$ and consider columnwise partitions for $\boldsymbol{V}$ and $\boldsymbol{C}$

$$\boldsymbol{V} = [v_1, \ldots, v_n], \qquad \boldsymbol{C} = [c_1, \ldots, c_n]$$

Then

$$\mathrm{Tr}(\mathbf{M}^{\mathrm{T}}\mathbf{R}) = \mathrm{Tr}\left(\sum_{i=1}^{n} \sigma_i v_i c_i^{\mathrm{T}}\right) - \sum_{i=1}^{n} \sigma_i v_i^{\mathrm{T}} c_i$$

Now $\boldsymbol{C}$ and $\boldsymbol{V}$ are both orthogonal, and then $\| c_i \| = \| v_i \| = 1$. On the hand, according to Cauchy-Schwartz, $(v_i^{\mathrm{T}} c_i)^2 \leq \| v_i \|^2 \| c_i \|^2 = 1$ and the equality holds for $v_i = c_i$ or $\boldsymbol{V} = \boldsymbol{C}$. Therefore, $\sum_{i=1}^{n} \sigma_i$ is an upper bound for $\mathrm{Tr}(\mathbf{M}^{\mathrm{T}}\mathbf{R})$ which is attained for $\mathbf{R} = \mathbf{U}\boldsymbol{V}^{\mathrm{T}}$. End of proof.

Remark 1. It is easy to see that the distance between $\mathbf{M}$ and $\mathbf{R}$ in metric (11) is given by $\sum_{i=1}^{n} (\sigma_i - 1)^2$, which is the standard way of describing how 'far' a matrix is from being orthogonal. A question that might be asked is what happens with the solution to the projection problem when the manifold $GL(n)$ is acted upon by the group $SO(n)$. The answer is given in the following proposition.

Proposition 2. The solution to the projection problem described above is both left and right invariant under actions of elements from $SO(n)$.

Proof. Let $\mathbf{M} \in GL(n), \mathbf{M} = \mathbf{U}\Sigma \boldsymbol{V}^{\mathrm{T}}$ and the corresponding projection $\mathbf{R} \in SO(n), \mathbf{R} = \mathbf{U}\boldsymbol{V}^{\mathrm{T}}$. Let any $\mathbf{L} \in SO(n)$ and $\bar{\mathbf{M}} = \mathbf{LM}$. Then an SVD for $\bar{\mathbf{M}}$ can be found from the SVD for $\mathbf{M}$ in the form $\bar{\mathbf{M}} = (\mathbf{LU})\Sigma \boldsymbol{V}^{\mathrm{T}}$. Then, by proposition 1, the projection of $\bar{\mathbf{M}}$ is $\mathbf{R} = \mathbf{LU}\boldsymbol{V}^{\mathrm{T}} = \mathbf{LR}$, which proves left invariance. Similarly, if $\mathbf{M}$ is acted from the right by $\mathbf{L} \in SO(n)$, then $\bar{\mathbf{M}} = \mathbf{ML} = \mathbf{U}\Sigma(\boldsymbol{V}^{\mathrm{T}}\mathbf{L})$ projects to $\mathbf{R} = \mathbf{U}\boldsymbol{V}^{\mathrm{T}}\mathbf{L} = \mathbf{RL}$, which implies right invariance. End of proof.

It is worth noting that other projection methods do not exhibit bi-invariance. For instance, it is customary to find the projection $\mathbf{R} \in SO(n)$ by applying a Gram–Schmidt procedure (QR decomposition). In this case it is easy to see that the solution is left invariant, but in general it is not right invariant.

5 PROJECTION ON $SE(N)$

Similarly to the previous section, if a metric of form (13) is defined in $GA(n)$, the corresponding projection on $SE(n)$ can be found.

Proposition 1. Let $\mathbf{B} \in GA(n)$ with the following block partition

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{B}_1 \in GL(n), \qquad \mathbf{B}_2 \in \mathbb{R}^n$$

and $\mathbf{B}_1 = \mathbf{U}\Sigma \boldsymbol{V}^{\mathrm{T}}$ the singular value decomposition of $\mathbf{B}_1$. Then the projection of $\mathbf{B}$ on $SE(n)$ is given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{U}\boldsymbol{V}^{\mathrm{T}} & \mathbf{B}_2 \\ 0 & 1 \end{bmatrix} \in SE(n).$$

Proof. Let

$$\mathbf{A} = \begin{bmatrix} \mathbf{R} & \boldsymbol{d} \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{R} \in SO(n), \qquad \boldsymbol{d} \in \mathbb{R}^n$$

The problem to be solved can be formulated as follows:

$$\min_{\mathbf{A} \in SE(n)} \| \mathbf{B} - \mathbf{A} \|_{GA}^2$$

Then

$$\| \mathbf{B} - \mathbf{A} \|_{GA}^2 = \mathrm{Tr}[(\mathbf{B} - \mathbf{A})^{\mathrm{T}}(\mathbf{B} - \mathbf{A})]$$

$$= \mathrm{Tr}(\mathbf{B}^{\mathrm{T}}\mathbf{B}) - 2\mathrm{Tr}(\mathbf{B}^{\mathrm{T}}\mathbf{A}) + \mathrm{Tr}(\mathbf{A}^{\mathrm{T}}\mathbf{A})$$

The quantity $\mathbf{B}^{\mathrm{T}}\mathbf{B}$ is not involved in the optimization. The observation that

$$\mathrm{Tr}(\mathbf{B}^{\mathrm{T}}\mathbf{A}) = \mathrm{Tr}(\mathbf{B}_1^{\mathrm{T}}\mathbf{R}) + (\mathbf{B}_2^{\mathrm{T}}\boldsymbol{d} + 1),$$
$$\mathrm{Tr}(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = 4 + \boldsymbol{d}^t\boldsymbol{d}$$

separates the initial problem into two subproblems:

$$(a)\ \max_{\mathbf{R} \in SO(n)} \mathrm{Tr}(\mathbf{B}_1^{\mathrm{T}}\mathbf{R})$$

and

$$(b)\ \min_{\boldsymbol{d} \in \mathbb{R}^n} [-2\mathbf{B}_2^{\mathrm{T}}\boldsymbol{d} + \boldsymbol{d}^{\mathrm{T}}\boldsymbol{d}]$$

From proposition 1, the solution to subproblem (a) is $\mathbf{R} = \mathbf{U}\boldsymbol{V}^{\mathrm{T}}$. For the second subproblem, let

$$f : \mathbb{R}^n \to \mathbb{R}, \qquad f(x) = -2\mathbf{B}_2^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$$

The critical points of the scalar function f are given by

$$\nabla f(x) = -2\mathbf{B}_2 + 2x = 0 \Rightarrow x = \mathbf{B}_2$$

and the Hessian $\nabla^2 f(x) = 2\mathbf{I}$ is always positive definite. Therefore, the solution is $\boldsymbol{d} = \mathbf{B}_2$, which concludes the proof.

Similar to $SO(n)$, invariance properties are exhibited by the projection on $SE(n)$.

Proposition 2. The solution to the projection problem on $SE(n)$ is left invariant under actions of elements from $SE(n)$. The projection is bi-invariant under rotations.

Proof. Let

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ 0 & 1 \end{bmatrix} \in GA(n)$$

and define $\boldsymbol{A}$, $\boldsymbol{U}$, Σ and $\boldsymbol{V}$ such that

$$\mathbf{B}_1 = \mathbf{U}\Sigma \boldsymbol{V}^{\mathrm{T}}, \qquad \mathbf{A} = \begin{bmatrix} U\boldsymbol{V}^{\mathrm{T}} & \mathbf{B}_2 \\ 0 & 1 \end{bmatrix} \in SE(n)$$

Let

$$\boldsymbol{X} = \begin{bmatrix} \mathbf{R} & \boldsymbol{d} \\ 0 & 1 \end{bmatrix}$$

be an arbitrary element from $SE(n)$. Under left actions of $\boldsymbol{X}$, the solution pair becomes

$$\boldsymbol{X}\mathbf{B} = \begin{bmatrix} \mathbf{R}\mathbf{B}_1 & \mathbf{R}\mathbf{B}_2 + \boldsymbol{d} \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{X}\mathbf{A} = \begin{bmatrix} \mathbf{R}\mathbf{U}\boldsymbol{V}^{\mathrm{T}} & \mathbf{R}\mathbf{B}_2 + \boldsymbol{d} \\ 0 & 1 \end{bmatrix}$$

which proves left invariance of the projection. For the second part, note that the right translated solution pair is

$$\mathbf{B}\boldsymbol{X} = \begin{bmatrix} \mathbf{B}_1\mathbf{R} & \mathbf{B}_1\boldsymbol{d} + \mathbf{B}_2 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{A}\boldsymbol{X} = \begin{bmatrix} \mathbf{U}\boldsymbol{V}^{\mathrm{T}}\mathbf{R} & \mathbf{U}\boldsymbol{V}^{\mathrm{T}}\boldsymbol{d} + \mathbf{B}_2 \\ 0 & 1 \end{bmatrix}$$

It is easy to see that $\mathbf{B}_1\mathbf{R} = \mathbf{U}\Sigma\boldsymbol{V}^{\mathrm{T}}\mathbf{R}$. If only rotations ($\boldsymbol{d} = 0$) are taken into consideration, right invariance is proved.

6 GENERATING SMOOTH CURVES ON $SE(3)$

Based on the results from the previous sections, a procedure for generating near-optimal curves on $SE(3)$ follows: generate the curves in the ambient space and project them onto $SE(3)$. Owing to the fact that the defined metric in $GA(3)$ is the same at all points, the corresponding Christoffel symbols are all zero. Consequently, the optimal curves in the ambient space assume simple analytical forms (i.e geodesics—straight lines, minimum acceleration curves—cubic polynomial curves, minimum jerk curves—fifth-order polynomial curves, all parameterized by time). The resulting curve in $GA(3)$ is linear in the boundary conditions, and therefore left and right invariant. Recall that the projection procedure on $SE(3)$ is left invariant, and so is the overall procedure.

The focus is on $SO(3)$. Owing to the product structure of both $SE(3) = SO(3) \times \mathbb{R}^3$ and the metric $\langle , \rangle_{SE}$ for $a = 0$, all the results can straightforwardly be extended to $SE(3)$.

6.1 Geodesics on $SO(3)$

The problem to be solved is generating a geodesic $\mathbf{R}(t)$ between given end positions $\mathbf{R}_1 = \mathbf{R}(0)$ and $\mathbf{R}_2 = \mathbf{R}(1)$ on $SO(3)$. Without loss of generality, it is assumed that $\mathbf{R}_1 = \mathbf{I}$. Indeed, a geodesic between two arbitrary positions $\mathbf{R}_1$ and $\mathbf{R}_2$ is the geodesic between $\boldsymbol{I}$ and $\mathbf{R}_1^{-1}\mathbf{R}_2$ left translated by $\mathbf{R}_1$. Exponential coordinates σ_1, σ_2, σ_3 are considered as local parameterization of $SO(3)$. If $\mathbf{R}_2 = e^{\hat{\omega}0}$, then the geodesic is the exponential mapping of the uniformly parameterized segment passing through 0 and $\omega_0(\sigma(t) = \omega_0 t)$ from the exponential coordinates:

$$\mathbf{R}(t) = e^{\hat{\sigma}(t)} = e^{\hat{\omega}_0 t}$$

The geodesic in the ambient manifold $GL(3)$ satisfying the given boundary conditions on $SO(3)$ is

$$\mathbf{M}(t) = \mathbf{I} + (\mathbf{R}_2 - \mathbf{I})t, \qquad t \in [0, 1]$$

An analytical expression for the projection of an arbitrarily parameterized line in the ambient $GL(3)$ onto $SO(3)$ is derived, which will answer the following three questions:

1. Does the projection of a geodesic from $GL(3)$ follow the same path as the true geodesic on $SO(3)$? If the answer is yes, then question 2 makes sense.
2. Do the above two curves have the same parameterization?
3. If the answer is no, can one find an appropriate parameterization of the line in the ambient manifold so that the projection is identical to the true geodesic on $SO(3)$?

The following proposition is the key result of this section.

Proposition. Let $\mathbf{M}(t) = \mathbf{I} + (\mathbf{R}_2 - \mathbf{I})f(t), t \in [0, 1]$ be a line in $GL(3)$ with $\mathbf{R}_2 = e_0^{\hat{\omega}} \in SO(3)$ (f continuous, $f(0) = 0, f(1) = 1$). Then the projection of this line $\mathbf{R}^{\perp}(t)$ onto $SO(3)$ is the exponential mapping of a segment drawn between the origin and ω_0 in exponential coordinates parameterized by $\theta(t)$:

$$\begin{aligned} \mathbf{M}(t) = \mathbf{U}(t)\Sigma(t)\boldsymbol{V}^{\mathrm{T}}(t) \Rightarrow \mathbf{R}^{\perp}(t) = \mathbf{U}(t)\boldsymbol{V}^{\mathrm{T}}(t) \\ = e^{\hat{\omega}_0}\theta(t) \end{aligned} \tag{15}$$

$$\theta(t) = \frac{1}{\| \boldsymbol{\omega}_0 \|} \mathrm{a}\tan 2(1 - f(t) + f(t)\cos \| \boldsymbol{\omega}_0 \|, f(t) \sin \| \boldsymbol{\omega}_0 \|). \tag{16}$$

Proof. The SVD decomposition of $\mathbf{M}(t) = \mathbf{I} + (\mathbf{R}_2 - \mathbf{I})f(t) = \mathbf{U}(t)\boldsymbol{\Sigma}(t)\boldsymbol{V}(t)^{\mathrm{T}}$ is needed, where $\mathbf{R}_2 = \mathrm{e}^{\hat{\omega}_0}$ and $f(t)$ is a continuous function defined on [0, 1] satisfying $f(0) = 0, f(1) = 1$. The first observation is

$$\mathbf{M}^{\mathrm{T}}(t)\mathbf{M}(t) = \mathbf{I} - f(t)(1 - f(t)\mathbf{N},$$
$$\mathbf{N} = 2\mathbf{I} - \mathbf{R}_2 - \mathbf{R}_2^{\mathrm{T}}$$

The eigenstructure of the constant and symmetric matrix $\mathbf{N}$ completely determines the SVD of $\mathbf{M}(t)$. Because $\mathbf{N}$ is symmetric and real, its eigenvalues will be real and the corresponding eigenspaces orthogonal. Let λ_i, $\boldsymbol{v}_i$ be an eigenvalue–eigenvector pair of $\mathbf{N}$. Then,

$$\mathbf{N}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i \Rightarrow \mathbf{M}^{\mathrm{T}}(t)\mathbf{M}(t) = (1 - f(t)(1 - f(t))\lambda_i \boldsymbol{v}_i$$

Therefore, theoretically, the desired SVD decomposition is determined at this moment:

1. The matrix $\mathbf{V}(t)$ can be chosen as a constant of the form $\mathbf{V} = [\boldsymbol{v}_1 \boldsymbol{v}_2 \boldsymbol{v}_3]$, where $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are orthonormal eigenvectors of $\mathbf{N}$.
2. The singular values are given by $s_i^2(t) = 1 - f(t) \times (1 - f(t))\lambda_i$ (it will be shown shortly that the right-hand side of this equality is always positive).
3. The time dependence of the projection will be contained in

$$\mathbf{U}(t) = [\boldsymbol{u}_1(t)\boldsymbol{u}_2(t)\boldsymbol{u}_3(t)], \qquad \boldsymbol{u}_i(t) = \frac{\mathbf{M}(t)\boldsymbol{v}_i}{s_i},$$
$$i = 1, 2, 3$$

Using the Rodrigues formula for $\mathbf{R}_2 = \mathrm{e}^{\hat{\omega}_0}$, it is easy to see that

$$\mathbf{N} = \frac{1 - \cos \| \boldsymbol{\omega}_0 \|}{\| \boldsymbol{\omega}_0 \|^2}(\hat{\omega}_0^2 + \hat{\omega}_0^{2\mathrm{T}})$$

from which it follows that the eigenvalues of $\mathbf{N}$ are given by

$$\lambda(\mathbf{N}) = 0, 2(1 - \cos \| \boldsymbol{\omega}_0 \|), 2(1 - \cos \| \boldsymbol{\omega}_0 \|)$$

and a set of three orthonormal eigenvectors by

$$\left\{ \frac{\boldsymbol{\omega}_0}{\| \boldsymbol{\omega}_0 \|}, \frac{1}{\sqrt{\omega_3^2 + \omega_1^2}} \begin{bmatrix} -\omega_3 \\ 0 \\ \omega_1 \end{bmatrix}, \frac{1}{\sqrt{\omega_2^2\omega_1^2 + (\omega_3^2 + \omega_1^2)^2 + \omega_3^2\omega_2^2}} \begin{bmatrix} -\omega_2\omega_1 \\ \omega_3^2 + \omega_1^2 \\ -\omega_3\omega_2 \end{bmatrix} \right\}$$

where $\boldsymbol{\omega}_0 = [\omega_1\omega_2\omega_3]^{\mathrm{T}}$. With the eigenstructure of $\mathbf{N}$ determined, it is possible to write

$$\boldsymbol{\Sigma}(t) = \mathrm{diag}\{1, s(t), s(t)\},$$
$$s(t) = \sqrt{2(1 - \cos \| \boldsymbol{\omega}_0 \|)f^2(t) - 2(1 - \cos \| \boldsymbol{\omega}_0 \|)f(t) + 1} \tag{17}$$

where the binomial under the square root is always positive because it is positive at zero and $1 - \cos \| \boldsymbol{\omega}_0 \| \in (0, 2)$ gives a negative discriminant. Some straightforward but rather tedious calculation leads to

$$\mathbf{U}(t)\mathbf{V}^{\mathrm{T}} = \mathbf{I} + \frac{\hat{\omega}_0}{\| \boldsymbol{\omega}_0 \|}\gamma_2(t) + (1 - \gamma_1(t)) + \frac{\hat{\omega}_0^2}{\| \boldsymbol{\omega}_0 \|^2}$$

where

$$\gamma_1(t) = \frac{1 - f(t) + f(t) \cos \| \boldsymbol{\omega}_0 \|}{s(t)},$$
$$\gamma_2(t) = \frac{f(t) \sin \| \boldsymbol{\omega}_0 \|}{s(t)}$$

The discussion is restricted to $\| \boldsymbol{\omega}_0 \| \in (0, \pi)$ (in accordance with the exponential coordinates) which will give $\gamma_2(t) > 0$. Note that $\gamma_1^2(t) + \gamma_2^2(t) = 1$, so it is appropriate to define a function $\theta(t) \in (0, 1)$ so that

$$\gamma_1(t) = \cos(\| \boldsymbol{\omega}_0 \| \, \theta(t)), \qquad \gamma_2(t) = \sin(\| \boldsymbol{\omega}_0 \| \, \theta(t))$$

By use of the Rodrigues formula again,

$$\mathbf{U}(t)\mathbf{V}^{\mathrm{T}} = \mathrm{e}^{\hat{\omega}_0 \theta(t)}$$

so the projected line is the exponential mapping of a segment between the origin and $\boldsymbol{\omega}_0$ in exponential coordinates. The parameterization of the segment is given by

$$\theta(t) = \frac{1}{\| \boldsymbol{\omega}_0 \|} \mathrm{a}\tan 2(1 - f(t) + f(t) \cos \| \boldsymbol{\omega}_0 \|,$$
$$f(t) \sin \| \boldsymbol{\omega}_0 \|)$$

This is the end of the proof.

Note that the obtained parameterization $\theta(t)$ satisfies the boundary conditions $\theta(0) = 0, \theta(1) = 1$.

As a particular case of the above proposition for $f(t) = t$, the following corollary answers the first two questions at the beginning of this section.

Corollary 1. The true geodesic on $SO(3)$ and the projected geodesic from $GL(3)$ with ends on $SO(3)$ follow the same path on $SO(3)$ but with different parameterizations. The projected curve is the exponential mapping of the same segment from the exponential coordinates

$$\mathbf{R}^{\perp}(t) = \mathrm{e}^{\hat{\omega}_0 \theta(t)}$$

with the following parameterization

$$\theta(t) = \frac{1}{\| \boldsymbol{\omega}_0 \|} \mathrm{a}\tan 2(1 - t + t\cos \| \boldsymbol{\omega}_0 \|, t\sin \| \boldsymbol{\omega}_0 \|)$$

The derivative of the function $\theta(t)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta(t) = \frac{\sin \| \boldsymbol{\omega}_0 \|}{\| \boldsymbol{\omega}_0 \| s(t)}$$

where $s(t)$ is given by (17). Plots of the function $\theta(t)$ and its derivative are given in Fig. 2 for $t \in [0, 1]$ and the magnitude of the displacement on the manifold $\| \boldsymbol{\omega}_0 \| \in (0, \pi)$.

The conclusion is that, even though the line in $GL(3)$ is followed at constant velocity, the projected curve on the manifold has low speed at the beginning, attains its maximum in the middle and slows down as it approaches the end-point. The larger the displacement $\| \boldsymbol{\omega}_0 \|$, the larger the discrepancy in speeds. Also note that the middle of the line is projected into the middle of the true geodesic because $\theta(0.5) = 0.5$ [i.e the functions t and $\theta(t)$ are equal at $t = 0.5$]. This result has been stated in reference **[3]** in the context of unit quaternions as local parameters of $SO(3)$ (viewed as the unit sphere S^3 in the projective space $\mathbb{R}P^3$).

To answer the third question, it is necessary to find a parameterization $f(t)$ ($f(0) = 0, f(1) = 1$) of the line in $GL(3)$ with ends on $SO(3)$, which gives uniform parameterization t of the projected curve in exponential coordinates. The solution of the following equation in f

$$\mathrm{atan2}(1 - f(t) + f(t)\cos \| \boldsymbol{\omega}_0 \|,$$
$$f(t)\sin \| \boldsymbol{\omega}_0 \| = t$$

is of the form

$$f(t) = \frac{\sin(\| \boldsymbol{\omega}_0 \| t)}{\sin(\| \boldsymbol{\omega}_0 \| (1 - t)) + \sin(\| \boldsymbol{\omega}_0 \| t)}$$

The answer to the third question is stated in the following corollary.

Corollary 2. The true geodesic on $SO(3)$ starting at $\mathbf{I}$ and ending at $\mathbf{R}_2 = e^{\boldsymbol{\omega}_0}$ is the projection of the following line from the ambient manifold $GL(3)$:

$$\mathbf{M}(t) = \mathbf{I} + (\mathbf{R}_2 - \mathbf{I})f(t), \qquad t \in [0, 1],$$
$$f(t) = \frac{\sin(\| \boldsymbol{\omega}_0 \| t)}{\sin(\| \boldsymbol{\omega}_0 \| (1 - t)) + \sin(\| \boldsymbol{\omega}_0 \| t)}$$

Illustrative plots of $f(t)$ and its derivative are given in Fig. 3 for $t \in [0, 1]$ and different values of the displacement $\| \boldsymbol{\omega}_0 \| \in (0, \pi)$. As expected, to attain a uniform speed on $SO(3)$, the line in $GL(3)$ should be followed at high speed at the beginning, slowing down in the middle and accelerating again near the end-point.

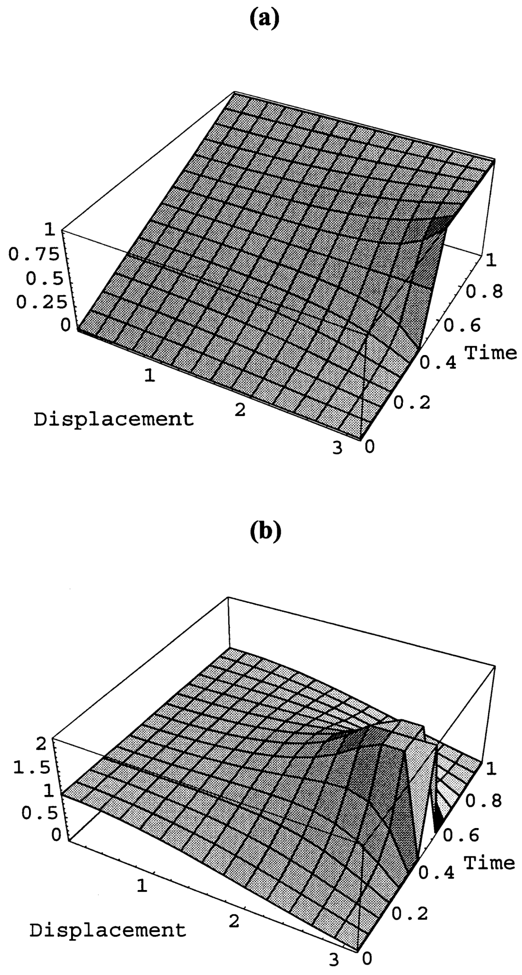

Fig. 2 (a) Function $\theta(t)$ and (b) the derivative $\mathrm{d}/\mathrm{d}t\theta(t)$

Remark. The result in corollary 2 is similar to the formula for spherical linear interpolation 'Slerp' in terms of quaternions **[3]**. The curve interpolating q_1 and q_2, with parameter u moving from 0 to 1, is given by

$$\mathrm{Slerp}(q_1, q_2; u) = \frac{\sin(1 - u)\theta}{\sin\theta} q_1 + \frac{\sin u\theta}{\sin\theta} q_2$$

where $q_1 \cdot q_2 = \cos\theta$.

6.2 Minimum acceleration curves on $SO(3)$

Firstly, the computation of optimal trajectories described in reference **[12]** is summarized. Then, near-optimal trajectories are generated via the projection method.

The necessary conditions for the curves that minimize the square of the L^2 norm of the acceleration are found by considering the first variation of the acceleration

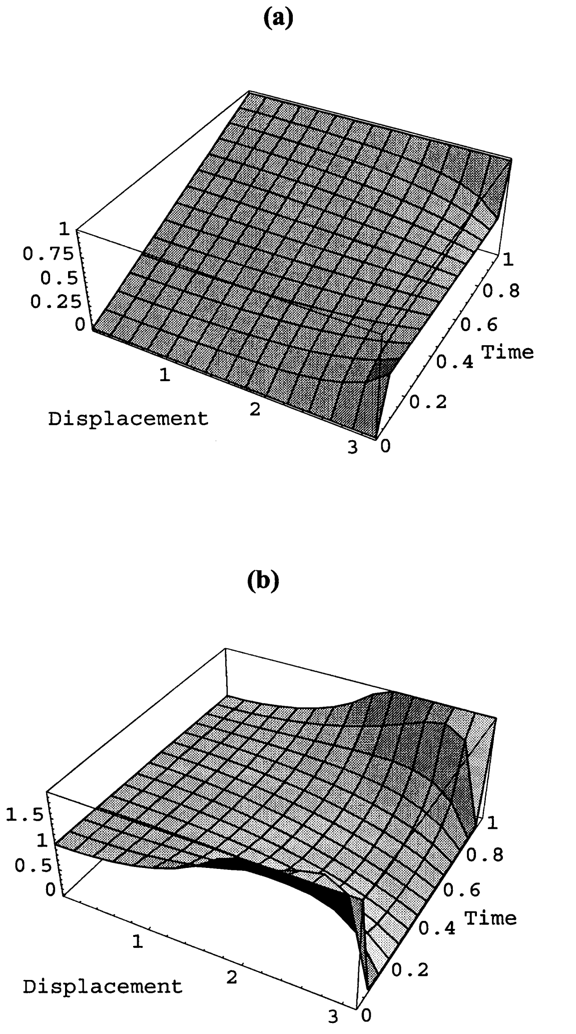

Fig. 3 (a) Function $f(t)$ and (b) the derivative $\mathrm{d}/\mathrm{d}tf(t)$

functional

$$L_a = \int_a^b \langle \nabla_V V, \nabla_V V \rangle \mathrm{d}t \tag{18}$$

where $V(t) = \frac{[\mathrm{d}\mathbf{R}(t)]}{\mathrm{d}t}$, ∇ is the symmetric affine connection compatible with a suitable Riemmanian metric and $\mathbf{R}(t)$ is a curve on the manifold. The initial and final points as well as the initial and final velocities for the motion are prescribed. The following result is stated and proved in reference [**12**].

Proposition. Let $\mathbf{R}(t)$ be a curve between two prescribed points on $SO(3)$ with metric $\langle , \rangle_{SO}$ that has prescribed initial and final velocities. If $\boldsymbol{\omega}$ is the vector from $so(3)$ corresponding to $V = \mathrm{d}\mathbf{R}/\mathrm{d}t$, the curve minimizes the cost function L_a derived from the canonical metric only if the following equation holds:

$$\boldsymbol{\omega}^{(3)} + \boldsymbol{\omega} \times \boldsymbol{\omega} = 0 \tag{19}$$

where $(\cdot)^{(n)}$ denotes the nth derivative of $(\cdot)$.

The above equation can be integrated to obtain

$$\boldsymbol{\omega}^{(2)} + \boldsymbol{\omega} \times \dot{\boldsymbol{\omega}} = \text{constant} \tag{20}$$

However, this equation cannot be further integrated analytically for arbitrary boundary conditions. In the special case where the initial and final velocities are tangential to the geodesic passing through the same points, the solution can be found by reparameterizing the geodesic [**12**]. In the general case, equation (19) must be solved numerically. A local parameterization of $SO(3)$ should be chosen, and three first-order differential equations will augment the system. The most convenient local coordinates are the exponential coordinates. Eventually, this ends up with solving a system of 12 first-order non-linear coupled differential equations with six boundary conditions at each end.

A solution can be found using iterative procedures such as the shooting method or the relaxation method. The latter has been chosen in this paper.

A more attractive and much simpler approach is the projection method described above. The main idea is to relax the problem to $GL(3)$, while keeping the proper boundary conditions on $SO(3)$ with the corresponding velocities. Minimum acceleration curves are found in $GL(3)$ and eventually projected back onto $SO(3)$.

In what follows, the time interval will be $t \in [0, 1]$ and the boundary conditions $\mathbf{R}(0)$, $\mathbf{R}(1)$, $\dot{\boldsymbol{R}}(0)$, $\dot{\boldsymbol{R}}(1)$ are assumed to be specified. The minimum acceleration curve in $GL(3)$ with a constant metric $\langle , \rangle_{GL}$ is a cubic given by

$$\mathbf{M}(t) = \mathbf{M}_0 + \mathbf{M}_1 t + \mathbf{M}_2 t^2 + \mathbf{M}_3 t^3$$

where $\mathbf{M}_0$, $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{M}_3 \in GL(3)$ are

$$\mathbf{M}_0 = \mathbf{R}(0), \qquad \mathbf{M}_1 = \dot{\boldsymbol{R}}(0)$$
$$\mathbf{M}_2 = -3\mathbf{R}(0) + 3\mathbf{R}(1) - 2\dot{\boldsymbol{R}}(0) - \dot{\boldsymbol{R}}(1)$$
$$\mathbf{M}_3 = 2\mathbf{R}(0) - 2\boldsymbol{R}(1) + \dot{\boldsymbol{R}}(0) + \dot{\boldsymbol{R}}(1)$$

Now the curve on $SO(3)$ is obtained by projecting $\mathbf{M}(t)$ onto $SO(3)$ as described in Section 4.

The following examples present comparisons between the minimum acceleration curves generated using the projection method and the curves obtained directly on $SO(3)$ by solving equations (19) using the relaxation method. All the generated curves are drawn in exponential local coordinates.

In Fig. 4, the following position boundary conditions were used:

$$\sigma(0) = [0\ 0\ 0]^{\mathrm{T}}, \qquad \sigma(1) = \left[\frac{\pi}{6}\ \frac{\pi}{3}\ \frac{\pi}{2}\right]^{\mathrm{T}} \tag{21}$$

The initial velocity is the one corresponding to the geodesic passing through the two positions:

$$\omega_0 = \left[\frac{\pi}{6}\ \frac{\pi}{3}\ \frac{\pi}{2}\right]^{\mathrm{T}}$$

Cases (a), (b) and (c) differ by the velocity at the end-point. Figure 4a corresponds to a final velocity $\omega_1 = \omega_0$, and therefore a minimum acceleration curve is obtained for which the end velocities are along the velocity of the corresponding geodesic, leading to a geodesic parameterized by a cubic of time [**12**]. The final velocity is $\omega_1 = \omega_0 + e_1$ in case (b) and $\omega_1 = \omega_0 + 5_{e_1}$ in case (c), where $e_1 = [1\ 0\ 0]^{\mathrm{T}}$.

As can be seen in Fig. 4a, the paths of the projected and the optimal curves are the same, the parameterizations are slightly different though, as expected. In cases (b) and (c), although the deviation of the final velocity from being homogeneous is large, the curves are close. Note that the boundary conditions are rigorously satisfied.

6.3 Motion generation on *SE*(3)

Since a method to generate (near) optimal curves on $SO(3)$ has been developed, the extension to $SE(3)$ is simply adding the well-known optimal curves from $\mathbb{R}^3$.

A homogeneous cubic rigid body is assumed to move (rotate and translate) in free space. The body frame $\{M\}$ is placed at the centre of mass and aligned with the principal axes of the body. A small square is drawn on one of its faces. The trajectory of the centre of the cube is starred.

The following boundary conditions were considered:

$$\sigma(0) = [0\ 0\ 0]^{\mathrm{T}}, \qquad \sigma(1) = \left[\frac{\pi}{6}\ \frac{2\pi}{3}\ \frac{\pi}{2}\right]^{\mathrm{T}}$$
$$\omega(0) = [1\ 2\ 3]^{\mathrm{T}}, \qquad \omega(1) = [2\ 1\ 1]^{\mathrm{T}}$$
$$\boldsymbol{d}(0) = [0\ 0\ 0]^{\mathrm{T}}, \qquad \boldsymbol{d}(1) = [8\ 10\ 12]^{\mathrm{T}}$$
$$\dot{\boldsymbol{d}}(0) = [1\ 1\ 1]^{\mathrm{T}}, \qquad \dot{\boldsymbol{d}}(1) = [1\ 5\ 3]^{\mathrm{T}}$$

True and projected minimum acceleration motions for a cubic rigid body with $a = 2$ and $m = 12$ are given in Fig. 5 for comparison.

In this example, the total displacement between initial and final positions on $SO(3)$ is large. If the rotational displacement is restricted near the origin of exponential coordinates, the simulated motions look identitical.

7 CONCLUSION

This paper has presented a method for generating smooth trajectories for a moving rigid body with specified boundary conditions. $SE(3)$, the set of all rigid body translations and orientations, was seen as a submanifold (and a subgroup) of the Lie group of affine maps in $\mathbb{R}^3$, $GA(3)$. The method involved two key steps:

(a) the generation of optimal trajectories in $GA(3)$,
(b) the projection of the trajectories from $GA(3)$ to $SE(3)$.

The overall procedure proved to be invariant with respect to both the local coordinates on the manifold and the choice of the inertial frame. The projected geodesic from $GL(3)$ and the actual geodesic on $SO(3)$ were shown to have identical paths on $SO(3)$. The

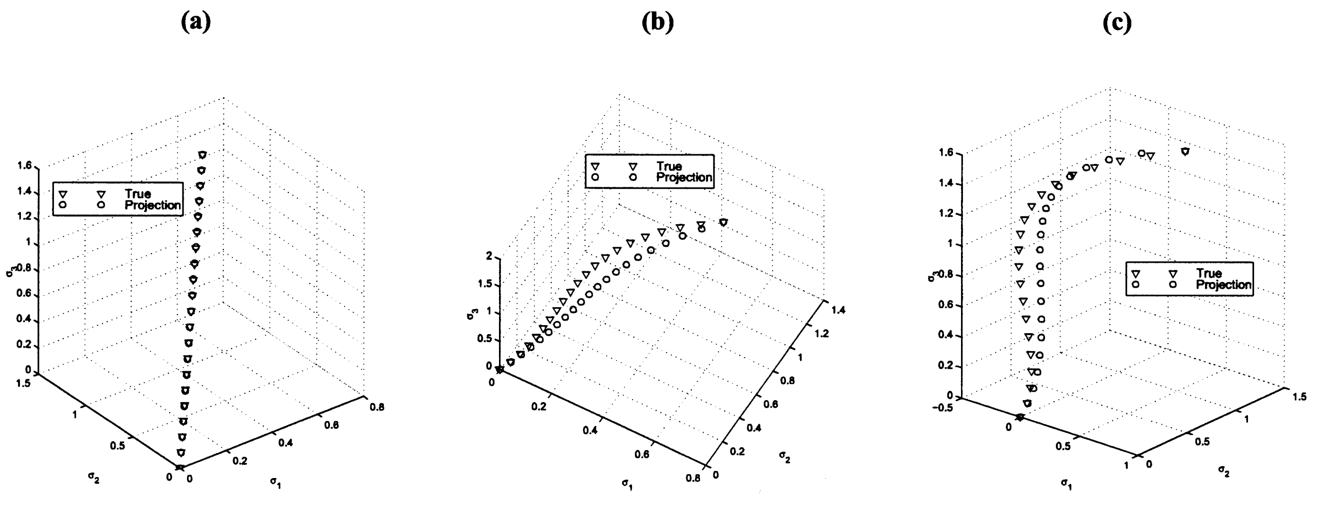

Fig. 4 Minimum acceleration curves on $SO(3)$ with a canonical metric: (a) velocity boundary conditions along the geodesic; (b) end velocity perturbed by e_1; (c) end velocity perturbed by $5e_1$

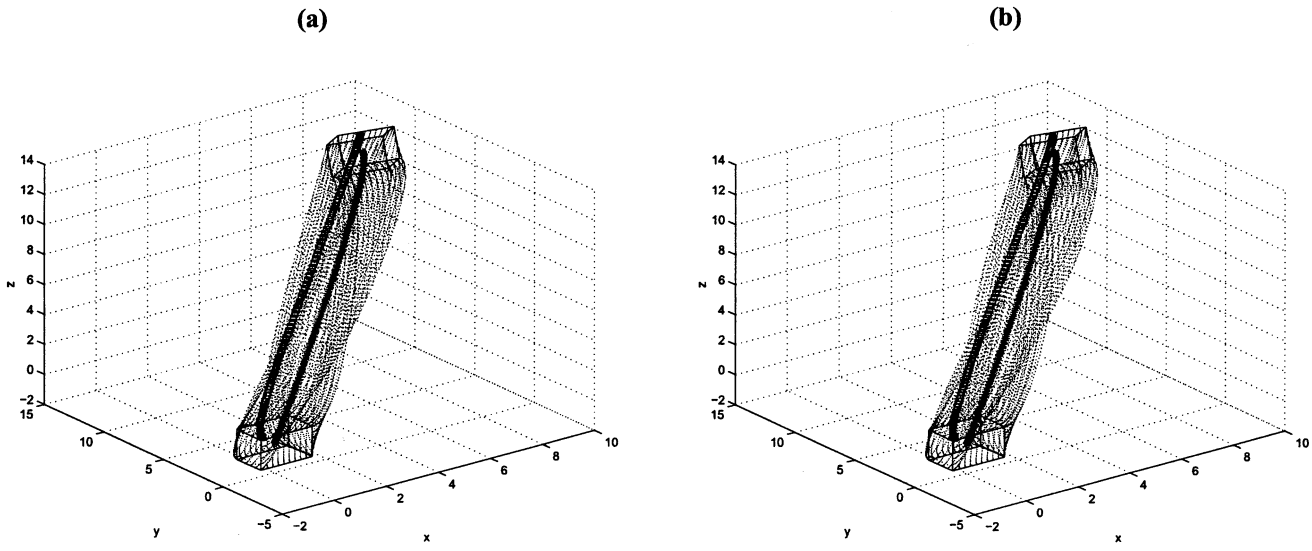

Fig. 5 Minimum acceleration motion for a cube in free space: (a) relaxation method; (b) projection method

parameterization of the line whose projection is the actual geodesic was derived.

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