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Brief Paper

Rotating stall control for axial flow compressors☆

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Abstract

Rotating stall is a primary constraint for the performance of axial flow compressors. This paper establishes a necessary and sufficient condition for a feedback controller to locally stabilize the critical equilibrium of the uniform flow at the inception of rotating stall. The explicit condition obtained in this paper provides an effective synthesis tool for rotating stall control.

Keywords: Compressors; Rotating stall; Hopf bifurcation; Hysteresis loops

Nomenclature

R	mean rotor radius
A_c	compressor duct area
U	blade speed at mean radius
a_s	speed of sound
B	$=(U/2a_s)\sqrt{V_p/(A_c Lc)}$
a, b	time-lag of blade passage
m	exit duct length factor
l_E, l_I, l_T	length of exit, entrance, throttle ducts, in wheel radius
l_c	total aerodynamic length
V_p	volume of plenum
p_S	static pressure in plenum
p_T	total pressure ahead of entrance and following the throttle duct
Ψ	$=(p_S - p_T)/\rho U^2$: pressure rise
ϕ	local flow coefficient at station 0
φ	disturbance flow at η
η	axial distance from station 0
Φ	mean flow coefficient at station 0
ξ	$=Ut/R$ with t time

1. Introduction

Axial flow compressors are subject to two distinct aerodynamic instabilities, rotating stall and surge, which can severely limit the compressor performance. Both these instabilities are disruptions of the normal operating condition which is designed for steady and axisymmetric flow. Rotating stall is a severely non-axisymmetric distribution of axial flow velocity, taking the form of a wave or "stall cell", that propagates steadily in the circumferential direction at a fraction of the rotor speed. Surge, on the other hand, is an axisymmetric oscillation of the mass flow along the axial length of the compressor. Both can result in catastrophic consequences.

Feedback control was proposed to improve compressor performance by Epstein, Ffowcs Williams, and Greitzer (1989) and has since received great attention in recent years. The existing results include linear control (Paduano, 1992), bifurcation stabilization (Liaw & Abed, 1996), and backstepping method (Krstic, Protz, Paduano, & Kokotovic, 1995; Banaszuk, Hauksson, & Mezic, 1996). This paper focuses on rotating stall control based on the multi-mode Moore–Greitzer model (Mansoux, Setiawan, Gysling, & Paduano, 1994), which in the limiting case converges to the full partial differential equation (PDE) model of Moore and Greitzer (1986). A feedback control law will be studied which is a generalization of the feedback control laws due to Liaw and Abed (1996), and Chen, Gu, Martin, and Zhou (1998). It will be shown that the critical operating point at the peak pressure rise of the performance characteristic curve is

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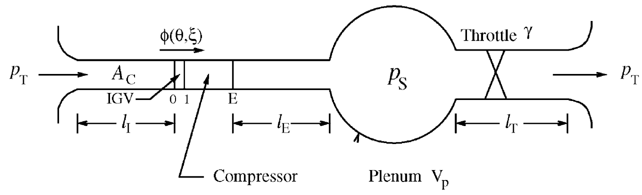

Fig. 1. Schematic of compressor showing nondimensionalized lengths.

locally stabilizable with the proposed feedback control law. A necessary and sufficient condition will be derived for local stabilization of the underlying feedback system. This condition holds, and results in lumped feedback controllers for the full PDE Moore–Greitzer model in the limiting case. In contrast to the global stabilization results of Banaszuk et al. (1996), our local stabilization condition is explicit, and provides an effective synthesis tool for rotating stall control without distributed sensors. Hence our result compliments those of Banaszuk et al. (1996).

2. Multi-mode compressor model and its equilibria

A schematic axial flow compressor is shown in Fig. 1. A post-stall model for compression systems was developed by Moore and Greitzer, and its full PDE form is described by Moore and Greitzer (1986)

$$\Psi + l_c \frac{d\Phi}{d\xi} = \psi_c(\phi) - m\left(\frac{\partial}{\partial \xi}\int_{-l_F}^{0} \varphi \, d\eta\right) - \left(\frac{1}{a}\frac{\partial \varphi}{\partial \xi} + \frac{1}{b}\frac{\partial \varphi}{\partial \theta}\right)\Bigg|_{\eta=0}, \tag{1}$$

$$\dot{\Psi} = \frac{1}{4B^2 l_c}(\Phi - \gamma\sqrt{\Psi}), \quad \phi = \Phi + \varphi|_{\eta=0}. \tag{2}$$

With the same assumption as in Moore and Greitzer (1986), the full PDE model can be approximated by (Mansoux et al., 1994)

$$\dot{\phi}_N = M_1\phi_N + M_2\psi_c(\Delta\phi_N) - M_2\bar{e}_N\Psi,$$
$$\Delta\phi_N = \frac{\phi_N}{W} - \bar{e}_N \in \mathbf{R}^{2N+1}, \tag{3}$$

$$\dot{\Psi} = \frac{1}{4B^2 l_c}\left(\frac{\bar{e}_N^{T}\phi_N}{2N+1} - \gamma\sqrt{\Psi}\right),$$
$$\bar{e}_N^{T} = [1 \quad 1 \quad \cdots \quad 1], \tag{4}$$

where, with $\{e^{j\theta_n}\}_{n=0}^{2N}$ as the set of $2N+1$ uniformly distributed samples on the unit circle,

$$\phi_N = \begin{bmatrix} \phi(\theta_1) \\ \phi(\theta_2) \\ \vdots \\ \phi(\theta_n) \\ \cdots \\ \phi(\theta_{2N+1}) \end{bmatrix},$$

$$T = \sqrt{\frac{2}{2N+1}}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & \dots & 1/\sqrt{2} \\ \cos\theta_1 & \cos\theta_2 & \dots & \cos\theta_{2N+1} \\ \sin\theta_1 & \sin\theta_2 & \dots & \sin\theta_{2N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \cos N\theta_1 & \cos N\theta_2 & \dots & \cos N\theta_{2N+1} \\ \sin N\theta_1 & \sin N\theta_2 & \dots & \sin N\theta_{2N+1} \end{bmatrix},$$

$$E = \mathrm{diag}(l_c, m_1, m_1, m_2, m_2, \dots, m_N, m_N),$$

$$m_n = \frac{m\cosh(nl_F)}{n\sinh(nl_F)} + \frac{1}{a},$$

$$F = \mathrm{diag}\left(0, \begin{bmatrix} 0 & -1/b \\ 1/b & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2/b \\ 2/b & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & -N/b \\ N/b & 0 \end{bmatrix}\right),$$

$$M_1 = T^{T}E^{-1}FT$$

and $M_2 = T^{T}E^{-1}T$. The multi-mode model as in (3) and (4) converges to the PDE model described in (1) and (2) uniformly as $N \to \infty$. It is worth to note that there hold

$$T^{-1} = T^{T}, \quad T\bar{e}_N = e_N\sqrt{2N+1},$$
$$T^{T}e_N = \frac{\bar{e}_N}{\sqrt{2N+1}}, \tag{5}$$

where $e_N^{T} = [1 \ 0 \ \cdots \ 0]$. Such a T is called discrete Fourier transform (DFT) matrix. The performance characteristic curve $\psi_c(\cdot)$ is assumed to be a cubic polynomial (Moore & Greitzer, 1986), and given by

$$\psi_c(\phi_N) = H[c_0\bar{e}_N + c_1\phi_N + c_3\phi_N^{\cdot 3}] \tag{6}$$

with $\phi_N^{\cdot k}$ raising each element of ϕ_N to the power of k. The uniform flow can be represented by $\phi_N = \phi_e\bar{e}_N$ with ϕ_e intensity of the flow rate.

Proposition 2.1. *The steady equilibria* $(\phi_N, \Psi) = (\phi_e\bar{e}_N, \Psi_e)$ *for the multi-mode compressor model in* (3) *and* (4) *are dependent on the throttle parameter* γ, *and governed by*

$$\Psi_e = H\left[c_0 + c_1\left(\frac{\phi_e}{W} - 1\right) + c_3\left(\frac{\phi_e}{W} - 1\right)^3\right] = \frac{\phi_e^2}{\gamma^2}. \tag{7}$$

The first equality in (7) is obtained by setting $\dot{\phi}_N = 0$ in (3), while the second equality in (7) by setting $\dot{\Psi} = 0$ in (4). A local model around equilibria $(\phi_e \bar{e}_N, \Psi_e)$ has the form

$$\dot{x} = f(\gamma, x) = Lx + Q[x,x] + C[x,x,x] + \cdots, \tag{8}$$

where $x = \delta\phi_N \oplus \delta\Psi$ with $\delta\phi_N = \phi_N - \phi_e \bar{e}_N$, $\delta\Psi = \Psi - \Psi_e$, and

$$L = S^{-1}\begin{bmatrix} \tau & \kappa e_N^{\mathrm{T}} \\ -\kappa e_N & E^{-1/2}(F + \alpha I)E^{-1/2} \end{bmatrix} S, \tag{9}$$

$$S = \begin{bmatrix} 0 & 2B\sqrt{(2N+1)l_c} \\ E^{1/2}T & 0 \end{bmatrix},$$

$$Q[x,x] = \begin{bmatrix} \dfrac{3Hc_3}{W^2}\left(\dfrac{\phi_e}{W} - 1\right)M_2 & 0_{2N+1} \\ 0_{2N+1}^{\mathrm{T}} & -\dfrac{\tau}{4\Psi_e} \end{bmatrix} x^{\cdot 2}, \tag{10}$$

$$C[x,x,x] = \begin{bmatrix} \dfrac{Hc_3}{W^3}M_2 & 0_{2N+1} \\ 0_{2N+1}^{\mathrm{T}} & \dfrac{\tau}{8\Psi_e^2} \end{bmatrix} x^{\cdot 3}, \tag{11}$$

$$\kappa = \frac{1}{2Bl_c}, \quad \tau = -\frac{\gamma}{8B^2 l_c \sqrt{\Psi_e}}, \tag{12}$$

$$\alpha = H\psi_c' = H\frac{\mathrm{d}\psi_c}{\mathrm{d}\phi_e} = \frac{H}{W}\left[c_1 + 3c_3\left(\frac{\phi_e}{W} - 1\right)^2\right]. \tag{13}$$

While κ is independent of the throttle parameter, τ and α are functions of γ. By the form of e_N, the linear matrix can be written as

$$L = S^{-1}DS, \quad D = \mathrm{diag}(D_0, D_1, \ldots, D_N), \tag{14}$$

$$D_0 = \begin{bmatrix} \tau & \kappa \\ -\kappa & \alpha/l_c \end{bmatrix}, \quad D_n = \begin{bmatrix} \alpha & -n/b \\ n/b & \alpha \end{bmatrix} m_n^{-1}, \tag{15}$$

where $1 \leqslant n \leqslant N$. The eigenvalues of L can now be easily computed as

$$\lambda_{1,2} = 0.5(\tau + \alpha/l_c) \pm 0.5\mathrm{j}\sqrt{4(\kappa^2 + \tau\alpha/l_c) - (\tau + \alpha/l_c)^2}, \tag{16}$$

$$\lambda_{2n+1,2n+2} = \left(\alpha \pm \mathrm{j}\frac{n}{b}\right)m_n^{-1}, \quad m_n = \frac{m\cosh(nl_F)}{n\sinh(nl_F)} + \frac{1}{a}, \tag{17}$$

where again $1 \leqslant n \leqslant N$. Hence, eigenvalues are also functions of γ.

Assume S-shape for the performance characteristic curve $\psi_c(\cdot)$, and $c_1 + 3c_3 = 0$ as in Moore and Greitzer (1986). A schematic compressor characteristic is shown in Fig. 2.

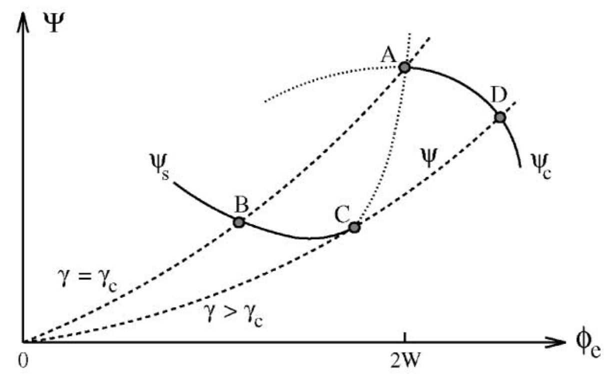

Fig. 2. Schematic compressor characteristic, showing rotating stall.

The designed operating point is uniquely determined by the intersection of the throttle line (dashed lines A–B or C–D for two different values of the throttle position) with the compressor performance curve $\psi_c(\cdot)$, according to (7). The maximum pressure rise takes place at $\alpha = 0$ (point A) because that is where the derivative of $\psi_c(\cdot)$ equals to zero. Since $\alpha < 0$ on the right side of the maximum pressure rise, the N pairs of eigenvalues of L matrix in (17) are stable. By $\tau < 0$ the pair of eigenvalues of (16) are also stable. Hence, any point on right side of the peak of $\psi_c(\cdot)$ is a stable operating point, and the uniform flow is a stable equilibrium, as highlighted by the solid line. However if the throttle value γ decreases, then the flow rate intensity ϕ_e decreases, and the derivative of $\psi_c(\cdot)$ eventually changes its sign into positive. Thus left side of the peak point A on $\psi_c(\cdot)$ corresponds to unstable operating points, as indicated by dotted line. It is noted that the N pairs of eigenvalues of L in (17) become imaginary precisely at $\alpha = 0$. The critical throttle value at which the N pairs of eigenvalues cross the imaginary axis simultaneously is given by, due to assumption $c_1 + 3c_3 = 0$,

$$\gamma_c = \left.\frac{\phi_e}{\sqrt{\Psi_e}}\right|_{\phi_e = \phi_c, \Psi_e = \Psi_c} = \left.\frac{\phi_c}{\sqrt{\psi_c(\phi_c)}}\right|_{\phi_c = 2W} = \frac{2W}{\sqrt{H(c_0 + c_1 + c_3)}} \tag{18}$$

and $\lambda_{2n+1,2n+2} = \pm\mathrm{j}\omega_n$ with $\omega_n = (n/b)m_n^{-1}$ for $1 \leqslant n \leqslant N$. At the criticality $\gamma = \gamma_c$, Hopf bifurcations occur for the multi-mode model of the compression system in (3) and (4) that induce rotating stall. Furthermore if the underlying bifurcation is subcritical, or unstable, the A–C portion of the stall curve is unstable which is sketched for $N = 1$. Stall cells will be born at point A, and grow which will throttle the operating point from A to B quickly which is a stable operating point, although undesirable. There is a tremendous drop in both the pressure rise, and flow rate. Moreover, increasing the throttle position and flow rate at point B does not increase the pressure rise. Rather the pressure rise becomes even lower before it comes to point C at which it jumps back to (due to again loss of stability) the performance curve $\psi_c(\cdot)$. The hysteresis loop A–B–C–D is the main cause for loss of compressor performance, and the potential damage to aeroengines. Hence stabilization of the critical operating point, and elimination of the hysteresis loop are the key for performance improvement of compression systems which will be addressed in the next section.

3. Local stabilization with feedback control

In the process of compressor design, stability of the critical operating point cannot be predicted and thus corrected before the compressor is produced. Hence feedback control was proposed by Epstein et al. (1989) to suppress rotating stall. Our paper considers the use of bleedvalve as actuator for the compression system. It is noted that the throttle parameter is a composition of uncertainty γ_o, synthesized disturbance from inlet and combustion chamber, and the bleed valve position $\delta\gamma$ that can be employed as actuator. Assume that Ψ is measurable. Then the throttle parameter can be expressed as

$$\gamma = \gamma_o + \delta\gamma = \gamma_o - \frac{u}{\sqrt{\Psi}}, \tag{19}$$

where u is the feedback control law to be designed. Our objective is local (asymptotic) stabilization around the critical equilibrium of the uniform flow $(\phi_c \bar{e}_N, \Psi_c)$ for the closed-loop compression system. The compression control system can now be written as

$$\dot{x} = f(\gamma, x) = Lx + Q[x,x] + C[x,x,x] + \cdots + \Gamma u, \tag{20}$$

where $\Gamma = [0 \ \cdots \ 0 \ 1]^{\mathrm{T}}\kappa^2 l_c$, and Lx, $Q[x,x]$, $C[x,x,x]$ are linear, quadratic, cubic terms, respectively, as given in (9), (10), and (11) with γ replaced by γ_o. Applying similarity transform $z = Sx$ yields the linearized system

$$\dot{z} = Dz + S\Gamma u, \quad D = SLS^{-1} = \mathrm{diag}(D_0, D_1, \ldots, D_N)$$

and $S\Gamma = e_N \kappa\sqrt{(2N+1)l_c}$. Thus, the state variables corresponding to the subsystem of D_0 are deviations in pressure rise $\delta\Psi = \Psi - \Psi_c$, and in average flow rate $\delta\Phi = \Phi - \phi_c$ at the criticality, respectively. The two eigenvalues of D_0 as in (16) are associated with surge dynamics, and are linearly controllable. The state variables corresponding to subsystems D_1–D_N represent amplitudes of harmonics for the disturbance flow rate φ at station $\eta = 0$. Thus, the N pairs of eigenvalues for D_1–D_N are associated with the rotating stall dynamics, and are linearly uncontrollable. Consequently, the axisymmetric equilibria on the left of point A (the $\psi_c(\cdot)$ curve in Fig. 2) are not stabilizable, posing challenge to rotating stall control. For stabilization of Hopf bifurcation induced by a single pair of critical (uncontrollable) eigenvalues, Abed and Fu (1986) proposed a quadratic feedback control law and proved its effectiveness, while more recent results of Gu et al. (1999) showed that a Hopf bifurcation is stabilizable, if and only if there exist stabilizing linear and quadratic feedback control laws. Therefore, we are motivated to consider the following feedback control law:

$$u = K_\psi \delta\Psi + \sum_{i=1}^{2N+1} K_{Q_i} x_i^2, \quad \delta\Psi = x_{2N+2}. \tag{21}$$

It is noted that the pressure rise Ψ is a lumped parameter which requires only a pair of pressure transducers in its measurement. However, the average disturbance flow requires measurement of x_1–x_{2N+1}, and thus distributed sensors as $N \to \infty$. Hence, the linear feedback term involving the average disturbance flow is intentionally skipped. For the same reason, more general quadratic form is avoided in the proposed feedback control law. Define discrete Fourier transform

$$\hat{K}_{Q_n} = \sum_{i=1}^{2N+1} K_{Q_i} \mathrm{e}^{-\mathrm{j}n\theta_i}, \quad 0 \leqslant n \leqslant 2N. \tag{22}$$

It will be shown later that local stabilization requires that $\hat{K}_{Q_0} \neq 0$, rather than $K_{Q_i} \neq 0$ for all $i > 0$. Hence, distributed sensors are unnecessary in the limiting case $N \to \infty$.

For the feedback control law as in (21), the closed-loop compression system can now be expanded into the following Taylor series:

$$\dot{x} = L^* x + Q^*[x,x] + C^*[x,x,x] + \cdots, \qquad L^* = S^{-1} D^* S, \tag{23}$$

$$D^* = \begin{bmatrix} \tau + \kappa^2 l_c K_\Psi & \kappa e_N^{\mathrm{T}} \\ -\kappa e_N & E^{-1/2}(F + \alpha I)E^{-1/2} \end{bmatrix} = \mathrm{diag}(D_0^*, D_1^*, \ldots, D_N^*), \tag{24}$$

$$Q^*[x,x] = \begin{bmatrix} \dfrac{3Hc_3}{W^2}\left(\dfrac{\phi_e}{W} - 1\right)M_2 & 0_{2N+1} \\ \kappa^2 l_c K_Q & -\frac{\tau}{4\Psi_c} \end{bmatrix} x^{\cdot 2},$$

$$C^*[x,x,x] = C[x,x,x] \tag{25}$$

and $K_Q = [K_{Q_1} \ \ldots \ K_{Q_{2N+1}}]$. It is noted that

$$D_0^* = \begin{bmatrix} \tau + \kappa^2 l_c K_\Psi & \kappa \\ -\kappa & \dfrac{\alpha}{l_c} \end{bmatrix}, \quad D_n^* = D_n, \quad 1 \leqslant n \leqslant N, \tag{26}$$

where $\det(\lambda I - D_0^*)$ is the characteristic polynomial associated with the surge dynamics, and $\det(\lambda I - D_n^*)$

with the rotating stall dynamics for $n \geqslant 1$. By

$$\det(\lambda I - D_0^*) = \lambda^2 - \left(\frac{\alpha}{l_c} + \tau + \kappa^2 l_c K_\Psi\right)\lambda + \frac{\alpha}{l_c}(\tau + \kappa^2 l_c K_\Psi) + \kappa^2,$$

the following result can be easily shown.

Lemma 3.1. *Under the feedback control law* (21), *the surge dynamics is locally stable at the criticality $\gamma_o = \gamma_c$, if and only if*

$$K_\Psi < -\frac{\tau}{\kappa^2 l_c}\bigg|_{\gamma_o=\gamma_c} = \frac{\gamma_c}{2\sqrt{\Psi_c}}. \quad (27)$$

For convenience we denote

$$\Delta K_\Psi = K_\Psi - \frac{\gamma_c}{2\sqrt{\Psi_c}}.$$

Condition (27) is thus equivalent to $\Delta K_\Psi < 0$, and in this case,

$$\tau + \kappa^2 l_c K_\Psi = \kappa^2 l_c \Delta K_\Psi$$

by (12). To determine stabilization condition for the feedback control law in (21) using the projection and Lyapunov methods as outlined in Appendix A, several vector quantities need be computed first for the feedback compression system. Denote

$$\Theta_n^{\mathrm{T}} = [\mathrm{e}^{-\mathrm{j}n\theta_1} \quad \mathrm{e}^{-\mathrm{j}n\theta_2} \quad \cdots \quad \mathrm{e}^{-\mathrm{j}n\theta_{2N+1}}]^{\mathrm{T}},$$

$$p_n = \sqrt{\left(\frac{2}{2N+1}\right)\bigg/ m_n}. \quad (28)$$

Then the left and right eigenvectors of linear matrix L^* associated with rotating stall are given by

$$\ell_n = m_n r_n^{\mathrm{H}}, \quad r_n = p_n[1 \quad j]u_n[\Theta_n^{\mathrm{T}} \quad 0]^{\mathrm{T}}, \quad (29)$$

where $n = 1, \ldots, N$, and u_n is an arbitrary non-zero column vector of size 2. Moreover $\ell_n r_n = 1$, if and only if $\|u_n\| = \sqrt{u_n^{\mathrm{T}} u_n} = 1/\sqrt{2}$. By component-wise multiplication, one can obtain

$$r_n \cdot \bar{r}_n = \frac{p_n^2}{2}\begin{bmatrix} \bar{e}_N \\ 0 \end{bmatrix}, \quad r_n \cdot r_n = \frac{p_n^2}{2}\begin{bmatrix} \Theta_{2n} \\ 0 \end{bmatrix}\mathrm{e}^{2\mathrm{j}\delta_n},$$

$$\delta_n = \tan^{-1}\left(\frac{u_{n2}}{u_{n1}}\right),$$

$$r_n \cdot r_p = \frac{p_n p_p}{2}\begin{bmatrix} \Theta_{n+p} \\ 0 \end{bmatrix}\mathrm{e}^{\mathrm{j}(\delta_n + \delta_p)},$$

$$r_n \cdot \bar{r}_p = \frac{p_n p_p}{2}\begin{bmatrix} \Theta_{n-p} \\ 0 \end{bmatrix}\mathrm{e}^{\mathrm{j}(\delta_n - \delta_p)},$$

where u_{n1} and u_{n2} are the first and second elements of u_n, respectively. The following lemma is important in deriving the explicit stabilization condition for the multi-mode feedback compression system described in (23)–(26).

Lemma 3.2. *Denote μ_{n_l} and v_{n_l} as the corresponding solutions to* (A.8) *and* (A.9), *respectively, for the uncontrolled compression system, and $\mu_{n_l}^*$ and $v_{n_l}^*$ as the solutions to* (A.8) *and* (A.9), *respectively, for the controlled compression system* (*with k replaced by n, p by l*) *respectively. Then at the criticality $\gamma_o = \gamma_c$,*

$$\mu_{n_l}^* = \mu_{n_l} + \Delta\mu_{n_l}, \quad (30)$$

$$\mu_{n_l} = \frac{\mathrm{j}3Hc_3 p_n p_l \mathrm{e}^{\mathrm{j}(\delta_n - \delta_l)}}{4m_{|n-l|}W^2(\omega_{|n-l|} - \omega_n + \omega_l)}\left(\frac{\phi_c}{W} - 1\right)\begin{bmatrix} \Theta_{n-l} \\ 0 \end{bmatrix}, \quad (31)$$

$$\Delta\mu_{n_l} = \frac{\hat{K}_{Q_{n-l}}\kappa^2 l_c p_n p_l \mathrm{e}^{\mathrm{j}(\delta_n - \delta_l)}}{4(\kappa^2 - \mathrm{j}(\omega_n - \omega_l)\kappa^2 l_c \Delta K_\Psi - (\omega_n - \omega_l)^2)}\begin{bmatrix} -\bar{e}_N/l_c \\ 2\mathrm{j}(\omega_n - \omega_l) \end{bmatrix}, \quad (32)$$

$$v_{n_l}^* = v_{n_l} + \Delta v_{n_l}, \quad (33)$$

$$v_{n_l} = \frac{3Hc_3 p_n p_l \mathrm{j}\mathrm{e}^{\mathrm{j}(\delta_n + \delta_l)}}{4m_{n+l}W^2(\omega_{n+l} - \omega_n - \omega_l)}\left(\frac{\phi_c}{W} - 1\right)\begin{bmatrix} \Theta_{n+l} \\ 0 \end{bmatrix}, \quad (34)$$

$$\Delta v_{n_l} = \frac{\hat{K}_{Q_{n+l}}\kappa^2 l_c p_n p_l \mathrm{e}^{\mathrm{j}(\delta_n + \delta_l)}}{4(\kappa^2 - \mathrm{j}(\omega_n + \omega_l)\kappa^2 l_c \Delta K_\Psi - (\omega_n + \omega_l)^2)} \times \begin{bmatrix} -\bar{e}_N/l_c \\ \mathrm{j}(\omega_l + \omega_n) \end{bmatrix}, \quad (35)$$

where $n, l = 1, 2, \ldots, N$, $n \neq l$, and for $n + l > N$ in (33), *$(n+l)$ is replaced by $\eta = (2N+1) - n - l$, except Θ_{n+l} by $\Theta_{-\eta}$.*

Proof. Since the expressions of μ_{n_l} and v_{n_l} were derived in Gu and Sparks (1998), we need verify only (32) and (35). By (A.8) and (A.9), and expressions of $Q_0^*[r_n, \bar{r}_l]$ and $Q_0[r_n, \bar{r}_l]$,

$$\Delta\mu_{n_l} = \mu_{n_l}^* - \mu_{n_l} = \tfrac{1}{2}([\mathrm{j}(\omega_n - \omega_l)I - L_0^*]^{-1}Q_0^*[r_n, \bar{r}_l] - [\mathrm{j}(\omega_n - \omega_l)I - L_0]^{-1}Q_0[r_n, \bar{r}_l]).$$

After lengthy calculations, $\Delta\mu_{n_l}$ can be simplified into

$$\Delta\mu_{n_l} = \frac{\hat{K}_{Q_{n-l}} p_n p_l \mathrm{e}^{\mathrm{j}(\delta_n - \delta_l)}}{8B\sqrt{l_c}} S^{-1}[\mathrm{j}(\omega_n - \omega_l)I - D^*]^{-1}\begin{bmatrix} e_N \\ 0 \end{bmatrix}.$$

The expression in (32) can then be verified by substitution of D^* at $\gamma_o = \gamma_c$, and carrying out the remaining computations. Similar calculations lead to

$$\Delta v_{n_l} = v_{n_l}^* - v_{n_l} = \tfrac{1}{2}([\mathrm{j}(\omega_n + \omega_l)I - L_0^*]^{-1}Q_0^*[r_n, r_l] - [\mathrm{j}(\omega_n + \omega_l)I - L_0]^{-1}Q_0[r_n, r_l])$$

$$= \frac{\hat{K}_{Q_{n-l}} p_n p_l \mathrm{e}^{\mathrm{j}(\delta_n + \delta_l)}}{8B}\sqrt{\frac{2N+1}{l_c}}S^{-1} \times [\mathrm{j}(\omega_n + \omega_l)I - D^*]^{-1}\begin{bmatrix} e_N \\ 0 \end{bmatrix},$$

from which (35) can be concluded. □

Using the same procedure as in the proof of Lemma 3.2, the following result can also be obtained. Since the proof is similar, it is omitted.

Corollary 3.3. *Denote μ_n and v_n as the corresponding solutions to* (A.4) *for the uncontrolled compression system, and μ_n^* and v_n^* as the solutions to* (A.4) *for the controlled compression system* (*with k replaced by n*), *respectively. Then at the criticality $\gamma_o = \gamma_c$,*

$$\mu_n^* = \mu_n + \Delta\mu_n, \quad \mu_n = \frac{3Hc_3 p_n^2}{4W^2}\left(\frac{\phi_c}{W} - 1\right)\begin{bmatrix} \dfrac{\gamma_c}{2\sqrt{\Psi_c}}\bar{e}_N \\ 1 \end{bmatrix}, \tag{36}$$

$$\Delta\mu_n = -\frac{p_n^2}{4}\left(\hat{K}_{Q_o} + \frac{3Hc_3}{W^2}\left(\frac{\phi_c}{W} - 1\right)K_\Psi\right)\begin{bmatrix} \bar{e}_N \\ 0 \end{bmatrix}, \tag{37}$$

$$v_n^* = v_n + \Delta v_n,$$

$$v_n = \frac{3Hc_3 p_n^2 \mathrm{j}\mathrm{e}^{\mathrm{j}2\delta_n}}{4m_{2n}W^2(\omega_{2n} - 2\omega_n^2)}\left(\frac{\phi_c}{W} - 1\right)\begin{bmatrix} \Theta_{2n} \\ 0 \end{bmatrix}, \tag{38}$$

$$\Delta v_n = \frac{\hat{K}_{Q_{2n}}\kappa^2 l_c p_n^2 \mathrm{e}^{2\mathrm{j}\delta_n}}{4(\kappa^2 - 2\mathrm{j}\omega_n\kappa^2 l_c \Delta K_\Psi - 4\omega_n^2)}\begin{bmatrix} -\bar{e}_N/l_c \\ 2\mathrm{j}\omega_n \end{bmatrix}. \tag{39}$$

The main result of this section is the following.

Theorem 3.4. *Assume that $c_3 < 0$, $\phi_c > W$, and condition* (27) *is satisfied. Moreover, suppose that the imaginary part of the eigenvalues in* (17) *are locally center-symmetric in the sense of Lyapunov. Then the rotating stall dynamics is locally asymptotically stable at the criticality $\gamma_o = \gamma_c$ for the feedback compression system described in* (23)–(26), *if and only if*

$$\hat{K}_{Q_o} < \frac{1}{4W}\left(\frac{\phi_c}{W} - 1\right)^{-1} - \frac{3Hc_3}{W^2}\left(\frac{\phi_c}{W} - 1\right)\left(K_\Psi - \frac{\gamma_c}{2\sqrt{\Psi_c}}\right). \tag{40}$$

Proof. Since (27) is true, local asymptotic stability at the criticality $\gamma_o = \gamma_c$ is determined by local asymptotic stability of the rotating stall dynamics. Suppose that the rotating stall dynamics is locally asymptotically stable at the criticality $\gamma_o = \gamma_c$. Then the nth SCV for the controlled compression system, denoted by $\tilde{\lambda}_{2*}^{(n)}$, for each projected rotating stall dynamics along the nth eigenvector is negative for $1 \leqslant n \leqslant N$. By the fact that the last element of ℓ_n is zero for $n \geqslant 1$,

$$\ell_n Q_0^*[r_n, x] = \ell_n Q_0[r_n, x],$$
$$\ell_n Q_0^*[\bar{r}_n, x] = \ell_n Q_0[\bar{r}_n, x]. \tag{41}$$

It follows that

$$\tilde{\lambda}_{2*}^{(n)} = \tilde{\lambda}_2^{(n)} + \Delta\lambda_2^{(n)},$$

$$\Delta\lambda_2^{(n)} = 2\,\mathrm{Re}\{2\ell_n Q_0[r_n, \Delta\mu_n] + \ell_n Q_0[\bar{r}_n, \Delta v_n]\},$$

where $\tilde{\lambda}_2^{(n)}$ is the nth SCV for the uncontrolled compression system, and given by

$$\tilde{\lambda}_2^{(n)} = \frac{3Hc_3 p_n^2}{4m_n W^3}\left[1 + \frac{6Hc_3\gamma_c}{W\sqrt{\Psi_c}}\left(\frac{\phi_c}{W} - 1\right)^2\right].$$

Straightforward calculation reveals that

$$\ell_n Q_0[x, y] = \frac{3Hc_3}{m_n W^2}\left(\frac{\phi_c}{W} - 1\right)\ell_n(x \cdot y), \tag{42}$$

$$\ell_n C_0[x, y, z] = \frac{Hc_3}{m_n W^3}\ell_n(x \cdot y \cdot z). \tag{43}$$

By normalization, $\ell_n r_n = 1$, and $\ell_n \bar{r}_n = 0$, there hold

$$\ell_n(r_n \cdot \Delta\mu_n) = -\frac{p_n^2}{4}\left(\hat{K}_{Q_o} + \frac{3Hc_3}{W^2}\left(\frac{\phi_c}{W} - 1\right)K_\Psi\right)$$

and $\ell_n(\bar{r}_n \cdot \Delta v_n) = 0$. The above implies that

$$\Delta\lambda_2^{(n)} = -\frac{3Hc_3 p_n^2}{m_n W^2}\left(\frac{\phi_c}{W} - 1\right) \times \left(\hat{K}_{Q_o} + \frac{3Hc_3}{W^2}\left(\frac{\phi_c}{W} - 1\right)K_\Psi\right).$$

Hence for $1 \leqslant n \leqslant N$, the nth SCV is given by

$$\tilde{\lambda}_{2*}^{(n)} = \frac{3Hc_3 p_n^2}{4m_n W^3}\left[1 + \frac{6Hc_3\gamma_c}{W\sqrt{\Psi_c}}\left(\frac{\phi_c}{W} - 1\right)^2\right] - \frac{3Hc_3 p_n^2}{m_n W^2}\left(\frac{\phi_c}{W} - 1\right)\left[\hat{K}_{Q_o} + \frac{3Hc_3}{W^2}\left(\frac{\phi_c}{W} - 1\right)K_\Psi\right]$$

$$= \frac{3Hc_3 p_n^2}{4m_n W^3}\left[1 - 4\hat{K}_{Q_o}W\left(\frac{\phi_c}{W} - 1\right) - \frac{12Hc_3}{W}\left(\frac{\phi_c}{W} - 1\right)^2\left(K_\Psi - \frac{\gamma_c}{2\sqrt{\Psi_c}}\right)\right].$$

Thus $\tilde{\lambda}_{2*}^{(n)} < 0$ for $1 \leqslant n \leqslant N$ yields condition (40) by $c_3 < 0$. Conversely suppose that (40) is true, then $\tilde{\lambda}_{2*}^{(n)} < 0$

for $1 \leqslant n \leqslant N$, and thus

$$\chi^*_{nn} = 8\tilde{\lambda}^{(n)}_{2^*} < 0, \quad n = 1, 2, \ldots, N.$$

Moreover by formula (A.7) and the relation in (41),

$$\chi^*_{n_l} = \chi_{n_l} + \Delta\chi_{n_l},$$

$$\Delta\chi_{n_l} = 16\,\mathrm{Re}\{\ell_n(Q_0[r_n, \Delta\mu_l] + Q_0[\bar{r}_l, \Delta\mu_{n_l}] + Q_0[r_l, \Delta\nu_{n_l}])\}$$

where χ_{n_l} is the solution to (A.7) for the uncontrolled compression system, and given by

$$\chi_{n_l} = \frac{6Hc_3 p_l^2}{m_n W^3}\left[1 + \frac{3Hc_3\gamma_c}{W\sqrt{\Psi_c}}\left(\frac{\phi_c}{W} - 1\right)^2\right].$$

Direct computation with property $\ell_n r_n = 1$, and $\ell_n r_l = \ell_n \bar{r}_l = 0$ gives

$$\ell_n(r_n \cdot \Delta\mu_l) = -\frac{p_l^2}{4}\left(\hat{K}_{Q_0} + \frac{3Hc_3}{W^2}\left(\frac{\phi_c}{W} - 1\right)K_\Psi\right),$$

$$\ell_n(r_l \cdot \Delta\mu_{n_l}) = \ell_n(\bar{r}_l \cdot \Delta\nu_{n_l}) = 0.$$

Hence using the relation in (42), (43), and expression for $\delta\chi_{n_l}$ yields

$$\Delta\chi_{n_l} = -\frac{6Hc_3 p_l^2}{m_n W^2}\left(\frac{\phi_c}{W} - 1\right) \times\left(2\hat{K}_{Q_0}W + \frac{6Hc_3}{W}\left(\frac{\phi_c}{W} - 1\right)K_\Psi\right).$$

Hence by $\chi^*_{n_l} = \chi_{n_l} + \Delta\chi_{n_l}$, we obtain

$$\chi^*_{n_l} = \frac{6Hc_3 p_l^2}{m_n W^3}\left[1 - 2\hat{K}_{Q_0}W\left(\frac{\phi_c}{W} - 1\right) - \frac{6Hc_3}{W}\left(\frac{\phi_c}{W} - 1\right)^2\left(K_\Psi - \frac{\gamma_c}{2\sqrt{\Psi_c}}\right)\right].$$

Since $\phi_e > W > 0$, condition (40) implies that

$$\hat{K}_{Q_0} < \frac{1}{2W}\left(\frac{\phi_c}{W} - 1\right)^{-1} - \frac{3Hc_3}{W^2}\left(\frac{\phi_c}{W} - 1\right)\left(K_\Psi - \frac{\gamma_c}{2\sqrt{\Psi_c}}\right)$$

which in turn implies that $\chi^*_{n_l} < 0$ by $c_3 < 0$. In light of the Lyapunov method (Theorem A.3), the critical operating point at $\gamma_o = \gamma_c$ is locally asymptotically stable. □

The significance of Theorem 3.4 lies in the fact that the stabilizability condition (40) requires only the sum of all K_{Q_i}'s to satisfy a bound determined by various compressor parameters and linear feedback gain K_Ψ. As a result, distributed sensing for the flow rate ϕ_N is unnecessary in the limiting case $N \to \infty$. Moreover, precise information on the cubic performance characteristic curve, and on the critical operating condition is not required, provided that upper and lower bounds on c_3, H, W, ϕ_c, and $\gamma_c/\sqrt{\Psi_c}$ are available.

Remark 3.5. In Banaszuk et al. (1996), backstepping method was used to derive a global stabilization condition on a more general state feedback control law than (21). It was shown that any given operating point (or set point) $\gamma_o \geqslant \gamma_c$ can be globally stabilized. It should be clear that if the set point is chosen as at the criticality, local stability will be ensured for $\gamma_o = \gamma_c$. However, it requires distributed sensing as $N \to \infty$. Moreover, the stabilizability condition in Banaszuk et al. (1996) is less explicit than the one in (40).

The stabilization condition (40) also indicates that the critical operating point can be locally stabilized by using linear feedback control law, i.e., by setting $K_Q = 0$ in (21). The next result follows easily from Theorem 3.4, and thus the proof is skipped.

Corollary 3.6. *Assume that* $c_3 < 0$, $\phi_c > W$, *and condition* (27) *is satisfied. Moreover, suppose that the imaginary part of the eigenvalues in* (17) *are locally center-symmetric in the sense of Lyapunov. Then the rotating stall dynamics is locally asymptotically stable at the criticality* $\gamma_o = \gamma_c$ *under feedback control* $u = K_\Psi\delta\Psi$, *if and only if*

$$K_\Psi > \frac{\gamma_c}{2\sqrt{\Psi_c}} + \frac{W}{12Hc_3}\left(\frac{\phi_c}{W} - 1\right)^{-2}. \tag{44}$$

Remark 3.7. Feedback stabilization using only the measurement of pressure rise was considered in Chen et al. (1998) for the third order Moore–Greitzer model which is a crude approximation. Corollary 3.6 is surprising which indicates that linear control law is also effective in stabilization of the critical operating point for the multi-mode model, which in the limiting case converges to the PDE Moore–Greitzer model. It should be pointed that the same feedback control law was studied in Leonessa, Chellaboina, and Haddad (1997) as well. Using equilibria dependent feedback control, Leonessa et al. (1997) was able to obtain a global stabilization result for $\gamma_o > \gamma_c$. In light of Corollary 3.6, condition (44) needs be enforced at $\gamma_o = \gamma_c$. In combination with the result in Corollary 3.6, global stabilization results of Leonessa et al. (1997) can be claimed for $\gamma_o \geqslant \gamma_c$.

It is worth to emphasizing that by $c_3 < 0$, the set of stabilizing gain K_Ψ satisfying both (44) and (27) is nonempty. However, the linear feedback control law $u = K_\Psi\delta\Psi$ has very limited effect on rotating stall control, because in satisfying the stabilization condition (44), it leaves very small surge margin in lieu of (27) and thus surge can be easily induced near the criticality $\gamma = \gamma_c$ thereby complicating the rotating stall dynamics. To avoid surge from taking place near the criticality, it is suggested to employ the linear term in (21) to enlarge the surge margin.

4. Illustrative simulations and concluding remarks

A compressor model in Moore and Greitzer (1986) is used for numerical simulations. The parameters of the model are given by

$$m = 1.75, \quad H = 0.18, \quad W = 0.25, \quad B = 0.1, \quad a = 1/3.5,$$

$$c_0 = 8/3, \quad c_1 = 1.5, \quad c_3 = -0.5, \quad l_c = 8, \quad l_F = \infty.$$

A persistent nonaxisymmetric flow perturbation (about 5% of ϕ_c) is added in (3) on the right-hand side of the equation in the simulation. Hence as γ decreases to the critical value, the equilibrium point is throttled into stall rather quickly. Increasing the throttle value at this time does not bring the equilibrium point back to the performance characteristic curve $\psi_c(\cdot)$ until the throttle value reaches $\gamma = 1.447\gamma_c$. If afterwards the throttle position is decreased to $\gamma = \gamma_c$, then the equilibrium point will be brought back to the peak of $\psi_c(\cdot)$. Fig. 3(a) shows the hysteresis loop as described where the dashed line is the performance characteristic curve. If linear control law $u = K_\Psi \delta \Psi$ is used, then the critical peak equilibrium on $\psi_c(\cdot)$ is locally stabilized for $0.1473 < K_\Psi < 0.3788$. Unfortunately, the equilibrium point with γ_o smaller than the critical value is unstable. Thus a similar, though smaller, hysteresis loop is observed as shown in Fig. 3(b) where $K_\Psi = 0.2$ and the same non-axisymmetric disturbance are used. The throttle value γ_o at which the equilibrium point returns to the performance characteristic curve is about $\gamma_o = 1.33\gamma_c < 1.447\gamma_c$. Although it was demonstrated by Chen et al. (1998) that the same linear feedback control law eliminates the hysteresis loop for $N = 1$, but it fails to do so for $N > 1$. This simulation result indicates the fact that the stable peak equilibrium point, and the hysteresis loop can co-exist for multi-mode compressor models which does not occur for the single-mode ($N = 1$) Moore–Greitzer model.

Because the linear feedback control law stabilizes only the peak equilibrium point locally, but fails to eliminate the hysteresis loop for the multi-mode Moore–Greitzer model, quadratic feedback control law is employed. For $K_\Psi = 0$, the critical equilibrium is locally stabilized if and only if $\hat{K}_{Q_0} < -0.6364$ in light of (40). Two simulations are carried out: (i) $K_{Q_i} = -1$ for each i, and (ii) $K_{Q_{1,4,7}} = -3$ and $K_{Q_{2,3,5,6,8,9}} = 0$ where $2N + 1 = 9$ for $N = 4$. The use of quadratic control law not only stabilizes the peak operating point, but also eliminates the hysteresis loop as demonstrated in Fig. 4.

In this simulation, the throttle value γ_o changes in the range of $[0.94\gamma_c, \gamma_c]$, and the system does not undergo significant pressure drop and no hysteresis loop is present. Moreover, there is actually no difference between the two cases. This observation concurs with the theoretical result derived in this paper: just a few flow measurements are necessary since the stabilizing condition is in terms of the sum of the quadratic coefficients. It should also mentioned that as γ_o decreases further to $\gamma_o < 0.94\gamma_o$, surge takes place which is not shown in Fig. 3, but the time responses of the pressure rise and average flow rate are plotted in Fig. 5 that clearly indicate the elimination of the hysteresis loop, and the effectiveness of the proposed rotating stall control law.

In summary, the simulation results in this section illustrate the possibility for rotating stall control without measuring all $2N + 1$ local flow rates while eliminating the hysteresis loop. This result is important due to the difficulty in employing distributed sensors in the limiting case. Moreover, the results of this paper provide effective synthesis procedures for rotating stall control. The simulation results illustrate the difference between the single-mode ($N = 1$) and multi-mode ($N > 1$) Moore–Greitzer models for the control purpose. Due to the space limit, our results on stability analysis for simultaneous rotating

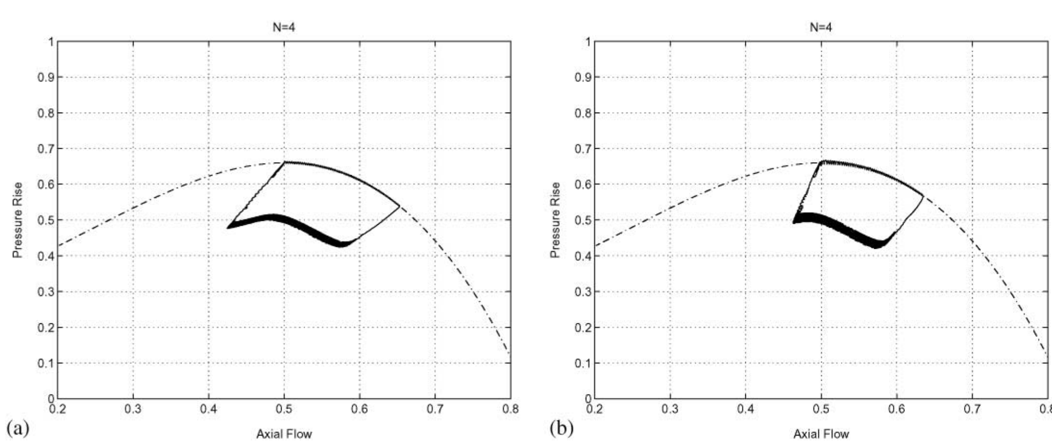

Fig. 3. Results for $N = 4$: pressure rise versus average flow: (a) hysteresis loop without control, (b) hysteresis loop with linear control.

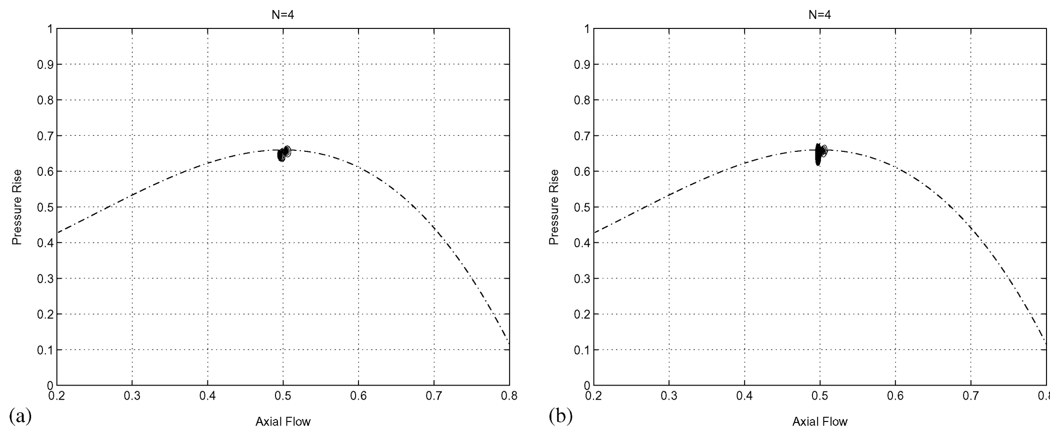

Fig. 4. Elimination of hysteresis loop with quadratic control: (a) all local flow rates are used, (b) Only three local flow rates are used.

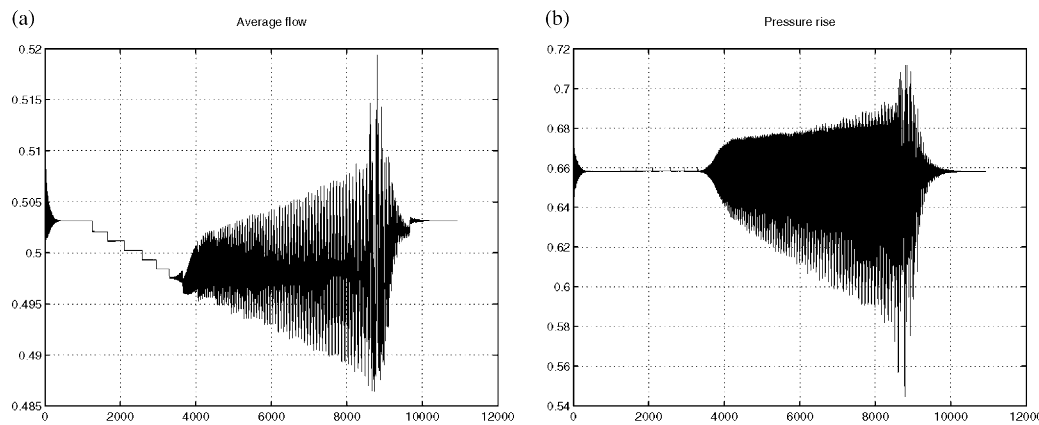

Fig. 5. Time responses with quadratic control: (a) time response of pressure rise, (b) time response of average flow rate.

stall and surge dynamics are not reported here. Interested readers are referred to Belta and Gu (1999) which is available upon request.

Appendix A. Stability conditions for Hopf bifurcation

Consider the following nth order parametrized nonlinear system:

$$\dot{x} = f(\gamma, x), \qquad f(\gamma, x_0) = 0 \quad \forall \gamma \in \mathbf{R},\ x_0 \in \mathscr{S} \subset \mathbf{R}^n, \quad \text{(A.1)}$$

where $x \in \mathbf{R}^n$, γ is a real-valued parameter, and $\mathscr{S}$ a linear subspace of $\mathbf{R}^n$, to be clarified later. It is assumed that $f(\cdot,\cdot)$ is sufficiently smooth such that the equilibrium solution x_e of $f(\gamma, x_e) = 0$ is a smooth function of γ. Thus,

$$\dot{x} = f(\gamma, x) = L(\gamma)x + Q(\gamma)[x,x] + C(\gamma)[x,x,x] + \cdots, \quad \text{(A.2)}$$

where $L(\gamma)x$, $Q(\gamma)[x,x]$, and $C(\gamma)[x,x,x]$ can each be expanded into

$$L(\gamma)x = L_0 x + \delta\gamma L_1 x + (\delta\gamma)^2 L_2 x + \cdots,$$

$$Q(\gamma)[x,x] = Q_0[x,x] + \delta\gamma Q_1[x,x] + \cdots,$$

$$C(\gamma)[x,x,x] = C_0[x,x,x] + \delta\gamma C_1[x,x,x] + \cdots$$

with $\delta\gamma = \gamma - \gamma_c$, and L_0, L_1, L_2 constant matrices of size $n \times n$. Suppose that $L(\gamma)$ possesses m ($< n/2$) pairs of complex eigenvalues $\lambda_k(\gamma) = \alpha_k(\gamma) \pm j\beta_k(\gamma)$, dependent smoothly on γ. It is assumed that for $1 \leqslant k \leqslant m$,

$$\alpha_k(\gamma_c) = 0, \quad \beta(\gamma_c) = \omega_k \neq 0, \qquad \alpha'_k(\gamma_c) = \frac{d\alpha_k}{d\gamma}(\gamma_c) \neq 0, \quad \text{(A.3)}$$

while all other eigenvalues of $L(\gamma)$ are stable at, and in a neighborhood of $\gamma = \gamma_c$. That is, m pairs of eigenvalues cross the imaginary axis *simultaneously*. Then each $\lambda_k(\gamma)$ is a critical eigenvalue, so is its conjugate. The center space, or the eigenspace for the m pairs of critical eigenvalues is denoted by $\mathscr{S}_c$, and is assumed to be orthogonally complement to $\mathscr{S}$ in the sense that $\mathscr{S} \oplus \mathscr{S}_c = \mathbf{R}^n$. Since projection of $x_0 \in \mathscr{S}$ to $\mathscr{S}_c$ is zero, the equilibria x_0 satisfying (A.1) will be called zero solution. The strict crossing assumption implies that the zero solution changes its stability as γ crosses γ_c. For instance, $\alpha'_k(\gamma_c) > 0$ implies that the zero solution is locally stable for $\gamma < \gamma_c$, and becomes unstable for $\gamma > \gamma_c$. The crucial problem is the determination of local stability near the critical parameter γ_c at which Hopf bifurcations occur, and the periodic solutions are born.

A.1. The projection method

The center space $\mathscr{S}_c$, spanned by all the critical eigenvectors, is complete. Denote $\mathscr{S}_{c_k}$ the eigenspace spanned by kth pair of the critical eigenvectors. If all the critical eigenvalues are distinct, then

$$\mathscr{S}_c = \mathscr{S}_{c_1} \oplus \mathscr{S}_{c_2} \oplus \cdots \oplus \mathscr{S}_{c_m}.$$

The projection method in Iooss and Joseph (1980) projects the non-linear dynamics into the subspace of $\mathscr{S}_{c_k}$ so that stability of the projected dynamics can be analyzed. Associated with each pair of the critical eigenvalues $(\lambda_k, \bar{\lambda}_k)$, there can define a stability characteristic value (SCV) $\tilde{\lambda}_2^{(k)}$ whose sign determines local stability of the kth projected dynamics. An algorithm to compute $\tilde{\lambda}_2^{(k)}$ is outlined next.

Step 1: Compute left row eigenvector ℓ_k and right column eigenvector r_k of L_0 corresponding to the kth critical eigenvalue of $\lambda_k(0) = \mathrm{j}\omega_k$. Normalize by setting $\ell_k r_k = 1$.

Step 2: Solve column vectors μ_k and v_k from the equations

$$-L_0\mu_k = \tfrac{1}{2}Q_0[r_k, \bar{r}_k], \quad (2j\omega_k I - L_0)v_k = \tfrac{1}{2}Q_0[r_k, r_k]. \tag{A.4}$$

Step 3: The coefficient $\tilde{\lambda}_2^{(k)}$ is given by

$$\tilde{\lambda}_2^{(k)} = 2\,\mathrm{Re}\{2\ell_k Q_0[r_k, \mu_k] + \ell_k Q_0[\bar{r}_k, v_k] + \tfrac{3}{4}\ell_k C_0[r_k, r_k, \bar{r}_k]\}. \tag{A.5}$$

Theorem A.1. *Suppose that* L_0 *has* m *pairs of nonzero critical eigenvalues on the imaginary axis with the rest on open left half-plane. Then the projected dynamics along the* k*th pair of the eigenvectors is stable, if* $\tilde{\lambda}_2^{(k)} < 0$, *and unstable if* $\tilde{\lambda}_2^{(k)} > 0$. *For the case of* $N = 1$, *the associated Hopf bifurcation is supercritical or stable, if* $\tilde{\lambda}_2 < 0$, *and subcritical or unstable, if* $\tilde{\lambda}_2 > 0$.

A.2. The Lyapunov method

The Lyapunov method for multiple critical modes is developed by Fu (1994). It gives sufficient conditions on the existence of a Lyapunov function that guarantees local stability of the bifurcated system. The following notion is needed.

Definition A.2. Under the condition that L_0 admits $N > 1$ pairs of eigenvalues on the imaginary axis with the rest on open left half-plane, the non-linear system (A.2) is said to be locally center-symmetric in the sense of Lyapunov if the following conditions:

$$w\omega_i \neq \omega_p, \quad \omega_i \neq 2\omega_p, \quad \omega_i \neq 3\omega_p,$$
$$m \geqslant 2,$$
$$\omega_i \neq \omega_p + \omega_k, \quad \omega_i \neq 2\omega_p + \omega_k, \quad 2\omega_i \neq \omega_p + \omega_k,$$
$$m \geqslant 3,$$
$$\omega_i \neq \omega_p + \omega_k + \omega_l, \quad \omega_i + \omega_p \neq \omega_k + \omega_l,$$
$$m \geqslant 4,$$

hold where all indices are distinct.

Theorem A.3. *Suppose that* L_0 *admits* $m > 1$ *pairs of eigenvalues on the imaginary axis with the rest on open left half-plane, and the non-linear system* (A.2) *is locally center-symmetric in the sense of Lyapunov. Then there exists a Lyapunov function that guarantees local stability of* (A.2) *at the criticality, if*

$$\chi_{kk} = 16\,\mathrm{Re}\{\ell_n(2Q_0[r_k, \mu_k] + Q_0[\bar{r}_k, v_k] + \tfrac{3}{4}C_0[r_k, r_k, \bar{r}_k])\} < 0 \tag{A.6}$$

for $k = 1, \ldots, m$, *and*

$$\chi_{kp} = 16\,\mathrm{Re}\{\ell_k(Q_0[r_k, \mu_p] + Q_0[\bar{r}_p, \mu_{k_p}] + Q_0[r_p, v_{k_p}] + \tfrac{3}{4}C_0[r_k, r_p, \bar{r}_p])\} \leqslant 0 \tag{A.7}$$

for $k, p = 1, \ldots, m$ *with* $k \neq p$, *where* μ_k, v_k *are the solutions to* (A.4), *and* μ_{k_p}, v_{k_p} *satisfy*

$$(\mathrm{j}(\omega_k - \omega_p)I_n - L_0))\mu_{k_p} = \tfrac{1}{2}Q_0[r_k, \bar{r}_p], \tag{A.8}$$

$$(\mathrm{j}(\omega_k + \omega_p)I_n - L_0)v_{k_p} = \tfrac{1}{2}Q_0[r_k, r_p]. \tag{A.9}$$

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