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Control Barrier Functions for Systems with High Relative Degree

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***Abstract*— This paper extends control barrier functions (CBFs) to high order control barrier functions (HOCBFs) that can be used for high relative degree constraints. The proposed HOCBFs are more general than recently proposed (exponential) HOCBFs. We introduce high order barrier functions (HOBFs), and show that their satisfaction of Lyapunov-like conditions implies the forward invariance of the intersection of a series of sets. We then introduce HOCBF, and show that any control input that satisfies the HOCBF constraint renders the intersection of a series of sets forward invariant. We formulate optimal control problems with constraints given by HOCBF and control Lyapunov functions (CLF), and provide a promising method to address the conflict between HOCBF constraints and control limitations by penalizing the class $\mathcal{K}$ functions. We illustrate the proposed method on an adaptive cruise control problem.**

I. INTRODUCTION

Barrier functions (BF) are Lyapunov-like functions [16][17], whose use can be traced back to optimization problems [5]. More recently, they have been employed in verification and control, e.g., to prove set invariance [4][14][18] and for multi-objective control [13]. Control BF (CBF) are extensions of BFs for control systems. Recently, it has been shown that CBF can be combined with control Lyapunov functions (CLF) [15][3][6][1] as constraints to form quadratic programs (QPs) [7] that are solved in real time. The CLF constraints can be relaxed [2] such that they do not conflict with the CBF constraints to form feasible QPs.

In [16], it was proved that if a barrier function for a given set satisfies Lyapunov-like conditions, then the set is forward invariant. A less restrictive form of a barrier function, which is allowed to grow when far away from the boundary of the set, was proposed in [2]. Another approach that allows a barrier function to be zero was proposed in [8] [11]. This simpler form has also been considered in time-varying cases and applied to enforce Signal Temporal Logic (STL) formulas as hard constraints [11].

The barrier functions from [2] and [8] work for constraints that have relative degree one (with respect to the dynamics of the system). A backstepping approach was introduced in [9] to address higher relative degree constraints, and it was shown to work for relative degree two. A CBF method for position-based constraints with relative degree two was also proposed in [19]. A more general form, which works for arbitrarily high relative degree constraints, was proposed in [12]. The method in [12] employs input-output linearization and finds a pole placement controller with negative poles to stabilize the barrier function to zero. Thus, this barrier function is an exponential barrier function.

In this paper, we propose a barrier function for high relative degree constraints, called high-order control barrier function (HOCBF), which is simpler and more general than the one from [12]. Our barrier functions are not restricted to exponential functions, and are determined by a set of class $\mathcal{K}$ functions. The general form of a barrier function proposed here is associated with the forward invariance of the intersection of a series of sets.

We formulate optimal control problems with constraints given by HOCBF and CLF and show that, by applying penalties on the class $\mathcal{K}$ functions, we can manage possible conflicts between HOCBF constraints and other constraints, such as control limitations. The main advantage of using the general form of HOCBF proposed in this paper is that it can be adapted to different types of systems and constraints.

We illustrate the proposed method on an adaptive cruise control problem. We consider linear and quadratic class $\mathcal{K}$ functions in the HOCBF. The simulations show that the results are heavily dependent on the choice of the class $\mathcal{K}$ functions.

II. PRELIMINARIES

The following lemma is used in the proof of this paper:

Lemma 1: (Lemma 4.4 in [10], Lemma 2.2 in [8]) Let $b : [t_0, t_f] \to \mathbb{R}$ be a continuously differentiable function (t_0, t_f denote the initial and final times, respectively). If $\dot{b}(t) \geq \alpha(b(t)), \forall t \in [t_0, t_f]$, where α is a class $\mathcal{K}$ function [10] of its argument, and $b(t_0) \geq 0$, then $b(t) \geq 0, \forall t \in [t_0, t_f]$.

Consider a system of the form

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}), \tag{1}$$

with $\boldsymbol{x} \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz. Solutions $\boldsymbol{x}(t)$ of (1), starting at $\boldsymbol{x}(t_0)$, $t \geq t_0$, are forward complete.

In this paper, we also consider affine control systems:

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u}, \tag{2}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, f is as defined above, $g : \mathbb{R}^n \to \mathbb{R}^{n \times q}$ is locally Lipschitz, and $\boldsymbol{u} \in U \subset \mathbb{R}^q$ (U denotes the control constraint set). Solutions $\boldsymbol{x}(t)$ of (2), starting at $\boldsymbol{x}(t_0), t \geq t_0$, are forward complete.

***Definition** 1:* A set $C \subset \mathbb{R}^n$ is forward invariant for system (1) (or (2)) if its solutions starting at any $\boldsymbol{x}(t_0) \in C$ satisfy $\boldsymbol{x}(t) \in C$, $\forall t \geq t_0$.

Let

$$C := \{\boldsymbol{x} \in \mathbb{R}^n : b(\boldsymbol{x}) \geq 0\}, \tag{3}$$

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where $b : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function.

***Definition** 2:* (*Barrier function* [8] [11]): The continuously differentiable function $b : \mathbb{R}^n \to \mathbb{R}$ is a barrier function (BF) for (1) if there exists a class $\mathcal{K}$ function α such that

$$\dot{b}(\boldsymbol{x}) + \alpha(b(\boldsymbol{x})) \geq 0, \quad (4)$$

for all $\boldsymbol{x} \in C$.

Theorem 1: ([8] [11]) Given a set C as in (3), if there exists a BF $b : C \to \mathbb{R}$, then C is forward invariant for (1).

***Definition** 3:* (*Control barrier function* [8] [11]): Given a set C as in Eqn. (3), $b(\boldsymbol{x})$ is a control barrier function (CBF) for system (2) if there exists a class $\mathcal{K}$ function α such that

$$L_f b(\boldsymbol{x}) + L_g b(\boldsymbol{x})\boldsymbol{u} + \alpha(b(\boldsymbol{x})) \geq 0 \quad (5)$$

for all $\boldsymbol{x} \in C$, where L_f, L_g denote the Lie derivatives along f and g, respectively.

Theorem 2: ([8] [11]) Given a CBF b with the associated set C from Eqn. (3), any Lipschitz continuous controller $\boldsymbol{u}(t) \in U, t \geq t_0$ that satisfies (5) renders the set C forward invariant for control system (2).

Remark 1: The barrier functions in Defs. 2 and 3 can be seen as more general forms of the ones in [2]. In particular, it can be shown that, for each barrier function defined in [2], we can always find one in the form of Def. 2.

***Definition** 4:* (*Control Lyapunov function* [1]) A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is a globally and exponentially stabilizing control Lyapunov function (CLF) for system (2) if there exist $c_1 > 0, c_2 > 0, c_3 > 0$ such that

$$c_1||\boldsymbol{x}||^2 \leq V(\boldsymbol{x}) \leq c_2||\boldsymbol{x}||^2 \quad (6)$$

$$\underset{u \in U}{inf}[L_f V(\boldsymbol{x}) + L_g V(\boldsymbol{x})\boldsymbol{u} + c_3 V(\boldsymbol{x})] \leq 0. \quad (7)$$

for $\forall \boldsymbol{x} \in \mathbb{R}^n$.

Theorem 3: ([1]) Given an exponentially stabilizing CLF V as in Def. 4, any Lipschitz continuous controller $\boldsymbol{u}(t)$ that satisfies (7) for all $t \geq t_0$ exponentially stabilizes system (2) to its zero dynamics (defined by the dynamics of the internal part if we transform the system to standard form and set the output to zero [10]).

***Definition** 5:* (*Relative degree* [10]) The relative degree of a continuously differentiable function $b : \mathbb{R}^n \to \mathbb{R}$ with respect to system (2) is the number of times we need to differentiate it along the dynamics of (2) until control $\boldsymbol{u}$ explicitly shows.

In this paper, since function b is used to define a constraint $b(\boldsymbol{x}) \geq 0$, we will also refer to the relative degree of b as the relative degree of the constraint.

Many existing work [2], [11], [12] combine CBF and CLF with quadratic costs to form optimization problems. Time is discretized and an optimization problem with constraints given by CBF and CLF is solved at each time step. Note that these constraints are linear in control since the state is fixed at the value at the beginning of the interval, and therefore the optimization problem is a quadratic program (QP). The optimal control obtained by solving the QP is applied at the current time step and held constant for the whole interval. The dynamics (2) is updated, and the procedure is repeated. It is important to note that this method works conditioned upon the fact that the control input shows up in (5), i.e., $L_g b(\boldsymbol{x}) \neq 0$.

III. HIGH ORDER CONTROL BARRIER FUNCTIONS

In this section, we define high order barrier functions (HOBF) and high order control barrier functions (HOCBF). We use a simple example to motivate the need for such functions and to illustrate the main ideas.

A. Example: Simplified Adaptive Cruise Control

Consider the simplified adaptive cruise control (SACC) problem[1] with the vehicle dynamics in the form:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} v(t) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad (8)$$

where $x(t)$ and $v(t)$ denote the position and velocity of the controlled vehicle along its lane, respectively, and $u(t)$ is its control.

We require that the distance between the controlled vehicle and its immediately preceding vehicle (the coordinates $x(t)$ and $x_p(t)$ of the preceding vehicle, respectively, are measured from the same origin and $x_p(t) \geq x(t), \forall t \geq t_0$) be greater than $\delta > 0$ for all the time, i.e.,

$$x_p(t) - x(t) \geq \delta, \forall t \geq t_0. \quad (9)$$

Assume the preceding vehicle runs at constant speed v_0. Let $\boldsymbol{x}(t) := (x(t), v(t))$ and $b(\boldsymbol{x}(t)) := x_p(t) - x(t) - \delta$, in order to use CBF (define $\alpha(\cdot)$ in Def. 3 as a linear function) to find control for the controlled vehicle such that the constraint (9) is satisfied, any control $u(t)$ should satisfy

$$\underbrace{v_0 - v(t)}_{L_f b(\boldsymbol{x}(t))} + \underbrace{0}_{L_g b(\boldsymbol{x}(t))} \times u(t) + \underbrace{x_p(t) - x(t) - \delta}_{b(\boldsymbol{x}(t))} \geq 0. \quad (10)$$

Notice that $L_g b(\boldsymbol{x}(t)) = 0$ in (10), so the control input $u(t)$ does not show up. Therefore, we cannot use these barrier functions to formulate an optimization problem as described at the end of Sec. II.

B. High Order Barrier Function (HOBF)

As in [11], we consider a time-varying function to define an invariant set for system (1). For a m^{th} order differentiable function $b : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}$, we define a series of functions $\psi_0 : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}, \psi_1 : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}, \psi_2 : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}, \dots, \psi_m : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}$ in the form:

$$\begin{aligned} \psi_0(\boldsymbol{x}, t) &:= b(\boldsymbol{x}, t) \\ \psi_1(\boldsymbol{x}, t) &:= \dot{\psi}_0(\boldsymbol{x}, t) + \alpha_1(\psi_0(\boldsymbol{x}, t)), \\ &\vdots \\ \psi_m(\boldsymbol{x}, t) &:= \dot{\psi}_{m-1}(\boldsymbol{x}, t) + \alpha_m(\psi_{m-1}(\boldsymbol{x}, t)), \end{aligned} \quad (11)$$

where $\alpha_1(\cdot), \alpha_2(\cdot), \dots, \alpha_m(\cdot)$ denote class $\mathcal{K}$ functions of their argument.

[1]A more realistic version of this problem, called the adaptive cruise control problem (ACC), is defined in Sec. IV.

We further define a series of sets $C_1(t), C_2(t), \ldots, C_m(t)$ associated with (11) in the form:

$$\begin{aligned} C_1(t) :=& \{\boldsymbol{x} \in \mathbb{R}^n : \psi_0(\boldsymbol{x}, t) \geq 0\} \\ C_2(t) :=& \{\boldsymbol{x} \in \mathbb{R}^n : \psi_1(\boldsymbol{x}, t) \geq 0\} \\ &\vdots \\ C_m(t) :=& \{\boldsymbol{x} \in \mathbb{R}^n : \psi_{m-1}(\boldsymbol{x}, t) \geq 0\} \end{aligned} \tag{12}$$

***Definition** 6:* Let $C_1(t), C_2(t), \ldots, C_m(t)$ be defined by (12) and $\psi_0(\boldsymbol{x}, t), \psi_1(\boldsymbol{x}, t), \ldots, \psi_m(\boldsymbol{x}, t)$ be defined by (11). A function $b : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}$ is a high order barrier function (HOBF) if it is m^{th} order differentiable for (1) and there exist differentiable class $\mathcal{K}$ functions $\alpha_1, \alpha_2 \ldots \alpha_m$ such that

$$\psi_m(\boldsymbol{x}, t) \geq 0 \tag{13}$$

for all $(\boldsymbol{x}, t) \in C_1(t) \cap C_2(t) \cap, \ldots, \cap C_m(t) \times [t_0, \infty)$.

Theorem 4: The set $C_1(t) \cap C_2(t) \cap, \ldots, \cap C_m(t)$ is forward invariant for system (1) if $b(\boldsymbol{x}, t)$ is a HOBF.

Proof: If $b(\boldsymbol{x}(t), t)$ is a HOBF, then $\psi_m(\boldsymbol{x}(t), t) \geq 0$ for $\forall t \in [t_0, \infty)$, i.e., $\dot{\psi}_{m-1}(\boldsymbol{x}(t), t) + \alpha_m(\psi_{m-1}(\boldsymbol{x}(t), t)) \geq 0$. By Lemma 1, since $\boldsymbol{x}(t_0) \in C_m(t_0)$ (i.e., $\psi_{m-1}(\boldsymbol{x}(t_0), t_0)) \geq 0$, and $\psi_{m-1}(\boldsymbol{x}(t), t)$ is an explicit form of $\psi_{m-1}(t)$), then $\psi_{m-1}(\boldsymbol{x}(t), t)) \geq 0$, $\forall t \in [t_0, \infty)$, i.e., $\dot{\psi}_{m-2}(\boldsymbol{x}(t), t) + \alpha_{m-1}(\psi_{m-2}(\boldsymbol{x}(t), t)) \geq 0$. Again, by Lemma 1, since $\boldsymbol{x}(t_0) \in C_{m-1}(t_0)$, we also have $\psi_{m-2}(\boldsymbol{x}(t), t)) \geq 0$, $\forall t \in [t_0, \infty)$. Iteratively, we can get $\boldsymbol{x}(t) \in C_i(t)$, $\forall i \in \{1, 2, \ldots, m\}, \forall t \in [t_0, \infty)$. Therefore, the set $C_1(t) \cap C_2(t) \cap, \ldots, \cap C_m(t)$ is forward invariant. ■

Remark 2: The sets $C_1(t), C_2(t), \ldots, C_m(t)$ should have a non-empty intersection at t_0 in order to satisfy the forward invariance condition starting from t_0 in Thm. 4. If $b(\boldsymbol{x}(t_0), t_0) \geq 0$, we can always choose proper class $\mathcal{K}$ functions $\alpha_1(\cdot), \alpha_2(\cdot), \ldots, \alpha_m(\cdot)$ to make $\psi_1(\boldsymbol{x}(t_0), t_0) \geq 0, \psi_2(\boldsymbol{x}(t_0), t_0) \geq 0, \ldots, \psi_{m-1}(\boldsymbol{x}(t_0), t_0) \geq 0$. There are some extreme cases, however, when this is not possible. For example, if $\psi_0(\boldsymbol{x}(t_0), t_0) = 0$ and $\dot{\psi}_0(\boldsymbol{x}(t_0), t_0) < 0$, then $\psi_1(\boldsymbol{x}(t_0), t_0)$ is always negative no matter how we choose $\alpha_1(\cdot)$. Similarly, if $\psi_0(\boldsymbol{x}(t_0), t_0) = 0$, $\dot{\psi}_0(\boldsymbol{x}(t_0), t_0) = 0$ and $\dot{\psi}_1(\boldsymbol{x}(t_0), t_0) < 0$, $\psi_2(\boldsymbol{x}(t_0), t_0)$ is also always negative, etc.. To deal with such extreme cases (as with the case when $b(\boldsymbol{x}(t_0), t_0) < 0$), we would need a feasibility enforcement method, which is beyond the scope of this paper.

C. High Order Control Barrier Function (HOCBF)

***Definition** 7:* Let $C_1(t), C_2(t), \ldots, C_m(t)$ be defined by (12) and $\psi_0(\boldsymbol{x}, t), \psi_1(\boldsymbol{x}, t), \ldots, \psi_m(\boldsymbol{x}, t)$ be defined by (11). A function $b : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}$ is a high order control barrier function (HOCBF) of relative degree m for system (2) if there exist differentiable class $\mathcal{K}$ functions $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that

$$\begin{aligned} L_f^m b(\boldsymbol{x}, t) + L_g L_f^{m-1} b(\boldsymbol{x}, t)\boldsymbol{u} + \frac{\partial^m b(\boldsymbol{x}, t)}{\partial t^m} \\ + O(b(\boldsymbol{x}, t)) + \alpha_m(\psi_{m-1}(\boldsymbol{x}, t)) \geq 0, \end{aligned} \tag{14}$$

for all $(\boldsymbol{x}, t) \in C_1(t) \cap C_2(t) \cap, \ldots, \cap C_m(t) \times [t_0, \infty)$. In the above equation, $L_g L_f^{m-1} b(\boldsymbol{x}, t) \in \mathbb{R}^{1 \times q}$, $O(\cdot)$ denotes the remaining Lie derivatives along f and partial derivatives with respect to t with degree less than or equal to $m-1$.

Theorem 5: Given a HOCBF $b(\boldsymbol{x}, t)$ from Def. 7 with the associated sets $C_1(t), C_2(t), \ldots, C_m(t)$ defined by (12), if $\boldsymbol{x}(t_0) \in C_1(t_0) \cap C_2(t_0) \cap, \ldots, \cap C_m(t_0)$, then any Lipschitz continuous controller $\boldsymbol{u}(t) \in U$ that satisfies (14) for all $t \geq t_0$ renders the set $C_1(t) \cap C_2(t) \cap, \ldots, \cap C_m(t)$ forward invariant for system (2).

Proof: Since $\boldsymbol{u}(t)$ is Lipschitz continuous and $\boldsymbol{u}(t)$ only shows up in the last equation of (11) when we take Lie derivative on (11), we have that $\psi_m(\boldsymbol{x}, t)$ is also Lipschitz continuous. The system state $\boldsymbol{x}$ in (2) are all continuously differentiable, so $\psi_1(\boldsymbol{x}, t), \psi_2(\boldsymbol{x}, t), \ldots, \psi_{m-1}(\boldsymbol{x}, t)$ are also continuously differentiable. Therefore, the HOCBF has the same property as the HOBF in Def. 6, and the proof is the same as the one for Theorem 4. ■

Note that, if we have a constraint $b(\boldsymbol{x}, t) \geq 0$ with relative degree m, then the number of sets is also m.

Remark 3: The general, time-varying HOCBF introduced in Def. 7, can be used for general, time-varying constraints (e.g., signal temporal logic specifications [11]) and systems. However, many problems, like the ACC problem that we consider in this paper, has time-invariant system dynamics and constraints. Therefore, in the rest of this paper, we focus on time-invariant versions for simplicity.

Remark 4: (*Relationship between time-invariant HOCBF and exponential CBF in* [12]) In Def. 6, if we set class $\mathcal{K}$ functions $\alpha_1, \alpha_2 \ldots \alpha_m$ to be linear functions with positive coefficients, then we can get exactly the same formulation as in [12] that is obtained through input-output linearization:

$$\begin{aligned} \psi_1(\boldsymbol{x}) :=& \dot{b}(\boldsymbol{x}) + k_1 b(\boldsymbol{x}) \\ \psi_2(\boldsymbol{x}) :=& \dot{\psi}_1(\boldsymbol{x}) + k_2 \psi_1(\boldsymbol{x}) \\ &\vdots \\ \psi_m(\boldsymbol{x}) :=& \dot{\psi}_{m-1}(\boldsymbol{x}) + k_m \psi_{m-1}(\boldsymbol{x}) \end{aligned} \tag{15}$$

where $k_1 > 0, k_2 > 0, \ldots, k_m > 0$. Therefore, the time-invariant version HOCBF defined in this paper is a generalization of the exponential CBF introduced in [12].

Example revisited. For the SACC problem introduced in Sec.III-A, the relative degree of the constraint from Eqn. (9) is 2. Therefore, we need a HOCBF with $m = 2$. We choose $\alpha_1(b(\boldsymbol{x}(t))) = b^2(\boldsymbol{x}(t))$ and $\alpha_2(\psi_1(\boldsymbol{x}(t))) = \psi_1^2(\boldsymbol{x}(t))$. In order for $b(\boldsymbol{x}(t)) := x_p(t) - x(t) - \delta$ to be a HOCBF for (8), a control input $u(t)$ should satisfy

$$\begin{aligned} L_f^2 b(\boldsymbol{x}(t)) + L_g L_f b(\boldsymbol{x}(t)) u(t) + 2b(\boldsymbol{x}(t)) L_f b(\boldsymbol{x}(t)) \\ + (L_f b(\boldsymbol{x}(t)))^2 + 2b^2(\boldsymbol{x}(t)) L_f b(\boldsymbol{x}(t)) + b^4(\boldsymbol{x}(t)) \geq 0. \end{aligned} \tag{16}$$

Note that $L_g L_f b(\boldsymbol{x}(t)) \neq 0$ in (16) and the initial conditions are $b(\boldsymbol{x}(t_0)) \geq 0$ and $\dot{b}(\boldsymbol{x}(t_0)) + b^2(\boldsymbol{x}(t_0)) \geq 0$.

D. Optimal Control for Time-Invariant Constraints

Consider an optimal control problem for system (2) with the cost defined as:

$$J(\boldsymbol{u}(t)) = \int_{t_0}^{t_f} \mathcal{C}(||\boldsymbol{u}(t)||) dt \tag{17}$$

where $||\cdot||$ denotes the 2-norm of a vector and $\mathcal{C}(\cdot)$ is a strictly increasing function of its argument. Assume a time-invariant (safety) constraint $b(\boldsymbol{x}) \geq 0$ with relative degree m has to be satisfied by system (2). Then $\boldsymbol{u}$ should satisfy the time-invariant HOCBF version of the constraint from (14):

$$-L_g L_f^{m-1} b(\boldsymbol{x})\boldsymbol{u} \leq L_f^m b(\boldsymbol{x}) + O(b(\boldsymbol{x})) + \alpha_m(\psi_{m-1}(\boldsymbol{x})) \tag{18}$$

for all $\boldsymbol{x} \in C_1 \cap C_2 \cap, \dots, \cap C_m$ ($C_i, i \in \{1, 2, \dots, m\}$ denotes the time-invariant version of $C_i(t)$).

If convergence to a given state is required in addition to optimality and safety, then, as in [2], HOCBF can be combined with CLF. We discretize the time and formulate a cost (17) while subjecting to the HOCBF constraint (18) and CLF constraint (7) at each time. With the optimal control input $\boldsymbol{u}$ obtained from (17) subject to (18), (7) and (2) at each time instant, we update the system dynamics (2) for each time step, and the procedure is repeated. Then the set $C_1 \cap C_2 \cap, \dots, \cap C_m$ is forward invariant, i.e., the safety constraint $b(\boldsymbol{x}(t)) \geq 0$ is satisfied for (2), $\forall t \in [t_0, t_f]$.

E. Feasibility of the Optimal Control Problem

In this section, we consider how we should properly choose class $\mathcal{K}$ functions $\alpha_1, \alpha_2, \dots, \alpha_m$ for a time-invariant HOCBF such that the feasibility of the optimal control problem defined in Sec. III-D is improved. For simplicity, in this section we assume that the term $L_g L_f^{m-1} b(\boldsymbol{x}(t))$ in (18) does not change sign for all $t \in [t_0, t_f]$.

Suppose we also have control limitations (componentwise inequality)

$$\boldsymbol{u}_{min} \leq \boldsymbol{u}(t) \leq \boldsymbol{u}_{max}, \forall t \in [t_0, t_f] \tag{19}$$

for system (2), $\boldsymbol{u}_{min}, \boldsymbol{u}_{max} \in \mathbb{R}^q$. If $L_g L_f^{m-1} b(\boldsymbol{x}) < \boldsymbol{0}$ ($\boldsymbol{0}$ denotes a zero vector of dim. q), we left product $-L_g L_f^{m-1} b(\boldsymbol{x})$ on (19):

$$\begin{aligned} -L_g L_f^{m-1} b(\boldsymbol{x}(t))\boldsymbol{u}_{min} &\leq -L_g L_f^{m-1} b(\boldsymbol{x}(t))\boldsymbol{u}(t) \\ &\leq -L_g L_f^{m-1} b(\boldsymbol{x}(t))\boldsymbol{u}_{max}, \forall t \in [t_0, t_f] \end{aligned} \tag{20}$$

The HOCBF constraint (18) may conflict with $-L_g L_f^{m-1} b(\boldsymbol{x}(t))\boldsymbol{u}_{min}$ in (20) (or $-L_g L_f^{m-1} b(\boldsymbol{x}(t))\boldsymbol{u}_{max}$ if $L_g L_f^{m-1} b(\boldsymbol{x}) > \boldsymbol{0}$). If this happens, the optimal control problem becomes infeasible. For the ACC problem defined in [2], this conflict is addressed by considering the minimum braking distance, which results in another complex constraint.

However, we may need to approximate the minimum braking distance with this method when we have non-linear dynamics and a cooperative optimization control problem [21][23]. This conflict is hard to address for high-dimensional systems. We can also explicitly consider all the constraints in finding the optimal analytical solutions [22], but limited to linear dynamics. Here, we discuss how we may deal with this conflict using the HOCBF introduced in this paper.

When (18) becomes active, its right hand side should be large enough such that (18) does not conflict with $-L_g L_f^{m-1} b(\boldsymbol{x}(t))\boldsymbol{u}_{min}$ in (20). We can add penalties $p_1 > 0, p_2 > 0, \dots, p_m > 0$ to address this conflict:

$$\begin{aligned} \psi_1(\boldsymbol{x}) &:= \dot{\psi}_0(\boldsymbol{x}) + p_1\alpha_1(\psi_0(\boldsymbol{x})) \\ \psi_2(\boldsymbol{x}) &:= \dot{\psi}_1(\boldsymbol{x}) + p_2\alpha_2(\psi_1(\boldsymbol{x})) \\ &\vdots \\ \psi_m(\boldsymbol{x}) &:= \dot{\psi}_{m-1}(\boldsymbol{x}) + p_m\alpha_m(\psi_{m-1}(\boldsymbol{x})) \end{aligned} \tag{21}$$

Remark 5: The penalties $p_1, p_2, \dots, p_m$ limit the demanding of the HOCBF constraint (18) for the control $\boldsymbol{u}$ after this constraint becomes active. This is helpful when we want to make the HOCBF constraint (18) comply with the control limitation by choosing small enough $p_1, p_2, \dots, p_m$, but the initial conditions should also be satisfied, i.e., $\boldsymbol{x}(t_0) \in C_1 \cap C_2 \cap, \dots, \cap C_m$.

IV. ACC PROBLEM FORMULATION

In this section, we consider a more realistic version of the adaptive cruise control (ACC) problem introduced in Sec.III-A, which was referred to as the simplified adaptive cruise control (SACC) problem. we consider that the safety constraint is critical and study the properties of HOCBF discussed in Sec.III-E.

Vehicle Dynamics : Instead of using the simple dynamics in (8), we consider more accurate vehicle dynamics in the form:

$$\underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{v}(t) \end{bmatrix}}_{\dot{\boldsymbol{x}}(t)} = \underbrace{\begin{bmatrix} v(t) \\ -\frac{1}{M}F_r(v(t)) \end{bmatrix}}_{f(\boldsymbol{x}(t))} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}}_{g(\boldsymbol{x}(t))} u(t) \tag{22}$$

where M denotes the mass of the controlled vehicle. $F_r(v(t))$ denotes the resistance force, which is expressed [10] as:

$$F_r(v(t)) = f_0 sgn(v(t)) + f_1 v(t) + f_2 v^2(t), \tag{23}$$

where $f_0 > 0, f_1 > 0$ and $f_2 > 0$ are scalars determined empirically. The first term in $F_r(v(t))$ denotes the coulomb friction force, the second term denotes the viscous friction force and the last term denotes the aerodynamic drag.

Constraint1 (Vehicle limitations): There are constraints on the speed and acceleration, i.e.,

$$\begin{aligned} v_{min} \leq v(t) \leq v_{max}, \forall t \in [t_0, t_f], \\ -c_d M g \leq u(t) \leq c_a M g, \forall t \in [t_0, t_f], \end{aligned} \tag{24}$$

where $v_{max} > 0$ and $v_{min} \geq 0$ denote the maximum and minimum allowed speeds, while $c_d > 0$ and $c_a > 0$ are deceleration and acceleration coefficients, respectively, and g is the gravity constant.

Constraint2 (Safety): Eqn. (9).

Objective1 (Desired Speed): The controlled vehicle always attempts to achieve a desired speed v_d.

Objective2 (Minimum Energy Consumption): We also want to minimize the energy consumption:

$$J(u(t)) = \int_{t_0}^{t_f} \left(\frac{u(t) - F_r(v(t))}{M}\right)^2 dt, \tag{25}$$

***Problem** 1:* Determine control laws to achieve Objectives 1, 2 subject to Constraints 1, 2, for the controlled vehicle governed by dynamics (22).

We use the HOCBF method to impose Constraints 1 and 2 on control input and a control Lyapunov function in Def. 4 to achieve Objective 1. We capture Objective 2 in the cost of the optimization problem.

V. ACC PROBLEM REFORMULATION

For ***Problem** 1*, we use the quadratic program (QP) - based method introduced in [2]. We consider two different types of class $\mathcal{K}$ functions (linear and quadratic functions) to define a HOCBF for Constraint 2.

A. Desired Speed (Objective 1)

We use a control Lyapunov function to stabilize $v(t)$ to v_d and relax the corresponding constraint (7) to make it a soft constraint [1]. Consider a Lyapunov function $V_{acc}(\boldsymbol{x}(t)) := (v(t)-v_d)^2$, with $c_1 = c_2 = 1$ and $c_3 = \epsilon > 0$ in Def. 4. Any control input $u(t)$ should satisfy

$$\begin{aligned}\underbrace{-\frac{2(v(t)-v_d)}{M}F_r(v(t))}_{L_f V_{acc}(\boldsymbol{x}(t))} + \underbrace{\epsilon(v(t)-v_d)^2}_{\epsilon V_{acc}(\boldsymbol{x}(t))} \\ + \underbrace{\frac{2(v(t)-v_d)}{M}}_{L_g V_{acc}(\boldsymbol{x}(t))} u(t) \leq \delta_{acc}(t)\end{aligned} \tag{26}$$

$\forall t \in [t_0, t_f]$. Here $\delta_{acc}(t)$ denotes a relaxation variable that makes (26) a soft constraint.

B. Vehicle Limitations (Constraint 1)

The relative degrees of speed limitations are 1, we use HOCBFs with $m = 1$ to map the limitations from speed $v(t)$ to control input $u(t)$. Let $b_1(\boldsymbol{x}(t)) := v_{max} - v(t)$, $b_2(\boldsymbol{x}(t)) := v(t) - v_{min}$ and choose $\alpha_1(b_i) = b_i, i \in \{1,2\}$ in Def. 7 for both HOCBFs. Then any control input $u(t)$ should satisfy

$$\underbrace{\frac{F_r(v(t))}{M}}_{L_f b_1(\boldsymbol{x}(t))} + \underbrace{\frac{-1}{M}}_{L_g b_1(\boldsymbol{x}(t))} u(t) + \underbrace{v_{max} - v(t)}_{b_1(\boldsymbol{x}(t))} \geq 0, \tag{27}$$

$$\underbrace{\frac{-F_r(v(t))}{M}}_{L_f b_2(\boldsymbol{x}(t))} + \underbrace{\frac{1}{M}}_{L_g b_2(\boldsymbol{x}(t))} u(t) + \underbrace{v(t) - v_{min}}_{b_2(\boldsymbol{x}(t))} \geq 0. \tag{28}$$

Since the control limitations are already constraints on control input, we do not need HOCBFs for them.

C. Safety Constraint (Constraint 2)

The relative degree of the safety constraint (9) is two. Therefore, we need to define a HOCBF with $m = 2$. Let $b(\boldsymbol{x}(t)) := x_p(t) - x(t) - \delta$, we consider two different forms of class $\mathcal{K}$ functions in Def. 7 (with a penalty $p > 0$ on both α_1, α_2 for all forms like (21)):

Form 1: Both α_1 and α_2 are linear:

$$\begin{aligned}\psi_1(\boldsymbol{x}(t)) &:= \dot{b}(\boldsymbol{x}(t)) + pb(\boldsymbol{x}(t)) \\ \psi_2(\boldsymbol{x}(t)) &:= \dot{\psi}_1(\boldsymbol{x}(t)) + p\psi_1(\boldsymbol{x}(t))\end{aligned} \tag{29}$$

Combining the dynamics (22) with (29), any control input $u(t)$ should satisfy

$$\begin{aligned}\underbrace{\frac{F_r(v(t))}{M}}_{L_f^2 b(\boldsymbol{x}(t))} + \underbrace{\frac{-1}{M}}_{L_g L_f b(\boldsymbol{x}(t))} u(t) + 2p\dot{b}(\boldsymbol{x}(t)) \\ + p^2 b(\boldsymbol{x}(t)) \geq 0.\end{aligned} \tag{30}$$

Form 2: Both α_1 and α_2 are quadratic:

$$\begin{aligned}\psi_1(\boldsymbol{x}(t)) &:= \dot{b}(\boldsymbol{x}(t)) + pb^2(\boldsymbol{x}(t)) \\ \psi_2(\boldsymbol{x}(t)) &:= \dot{\psi}_1(\boldsymbol{x}(t)) + p\psi_1^2(\boldsymbol{x}(t))\end{aligned} \tag{31}$$

Combining the dynamics (22) with (31), any control input $u(t)$ should satisfy

$$\begin{aligned}\underbrace{\frac{F_r(v(t))}{M}}_{L_f^2 b(\boldsymbol{x}(t))} + \underbrace{\frac{-1}{M}}_{L_g L_f b(\boldsymbol{x}(t))} u(t) + 2p\dot{b}(\boldsymbol{x}(t))b(\boldsymbol{x}(t)) \\ + p\dot{b}^2(\boldsymbol{x}(t)) + 2p^2\dot{b}(\boldsymbol{x}(t))b^2(\boldsymbol{x}(t)) + p^3 b^4(\boldsymbol{x}(t)) \geq 0.\end{aligned} \tag{32}$$

D. Reformulated ACC Problem

We partition the time interval $[t_0, t_f]$ into a set of equal time intervals $\{[t_0, t_0+\Delta t), [t_0+\Delta t, t_0+2\Delta t), \dots\}$, where $\Delta t > 0$. In each interval $[t_0 + \omega\Delta t, t_0 + (\omega+1)\Delta t)$ ($\omega = 0, 1, 2, \dots$), we assume the control is constant (i.e., the overall control will be piece-wise constant), and reformulate (approximately) ***Problem** 1* as a sequence of QPs. Specifically, at $t = t_0 + \omega\Delta t$ ($\omega = 0, 1, 2, \dots$), we solve

$$\boldsymbol{u}^*(t) = \arg\min_{\boldsymbol{u}(t)} \frac{1}{2}\boldsymbol{u}(t)^T H \boldsymbol{u}(t) + F^T \boldsymbol{u}(t) \tag{33}$$

$$\boldsymbol{u}(t) = \begin{bmatrix} u(t) \\ \delta_{acc}(t) \end{bmatrix}, H = \begin{bmatrix} \frac{2}{M^2} & 0 \\ 0 & 2p_{acc} \end{bmatrix}, F = \begin{bmatrix} \frac{-2F_r(v(t))}{M^2} \\ 0 \end{bmatrix}.$$

subject to the constraints from the maximum control constraint in Constraint 1, Eqns. (26), (27), (28) and (30) for Form 1 ((32) for Form 2), which are all linear in terms of controls in each interval. Their expressions are omitted due to space constraints, but is shown in [20]. We also assume F is a constant vector in each interval. $p_{acc} > 0$ is a penalty on the relaxation $\delta_{scc}(t)$. After solving (33), we update (22) with $u^*(t)$, $\forall t \in (t_0 + \omega\Delta t, t_0 + (\omega+1)\Delta t)$.

Remark 6: The minimum control constraint in Constraint 1 is not included in (33). This follows from Remark 5 in Sec.III-E that we may choose small enough p such that the minimum control constraint is satisfied.

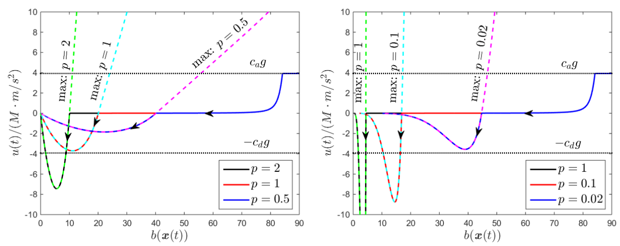
VI. IMPLEMENTATION AND RESULTS

In this section, we present case studies for ***Problem** 1* to illustrate the properties described in Sec.III-E. As noticed in (33), the term $L_g L_f b(\boldsymbol{x}(t)) = -\frac{1}{M}$ depends only on M. Therefore, the assumption in Sec. III-E is satisfied.

All simulations were conducted in MATLAB. We used quadprog to solve the QPs and ode45 to integrate the dynamics. The parameters are $v(t_0) = 20m/s$, $x_p(t_0) - x(t_0) = 100m$, $v_0(t) = 13.89m/s$, $v_d = 24m/s, \delta = 10m, M = 1650kg, g = 9.81m/s^2, f_0 = 0.1N, f_1 = 5Ns/m, f_2 =$

$0.25Ns^2/m, v_{max} = 30m/s, v_{min} = 0m/s, \Delta t = 0.1s, \epsilon = 10, c_a = c_d = 0.4, p_{acc} = 1$.

By Remark 5, we may find a small enough p in (29) and (31) such that the HOCBF constraints (30) and (32) do not conflict with the minimum control limitation. We present the case studies for the linear and quadratic class $\mathcal{K}$ functions in Fig.1(a) and Fig.1(b), respectively. The dashed lines in Fig. 1(a) denote the value of the right-hand side of the HOCBF constraint (i.e, $\frac{L_f^m b(\boldsymbol{x}) + O(b(\boldsymbol{x})) + \alpha_m(\psi_{m-1}(\boldsymbol{x}))}{-L_g L_f^{m-1} b(\boldsymbol{x})}$ in (18)), and the solid lines are the optimal controls obtained by solving (33). When the dashed lines and solid lines coincide, the HOCBF constraints (30) and (32) are active.



(a) Under linear class $\mathcal{K}$ function. (b) Under quadratic class $\mathcal{K}$ function.

Fig. 1. Control input $u(t)$ as $b(\boldsymbol{x}(t)) \to 0$ for different p values under different class $\mathcal{K}$ functions. The arrows denote the changing trend for $b(\boldsymbol{x}(t))$ with respect to time.

In Fig.1(a) and Fig.1(b), the HOCBF constraint does not conflict with the braking limitation when $p = 1$ and $p = 0.02$ for linear and quadratic class $\mathcal{K}$ functions, respectively. The minimum control input increases as p decreases.

Then, we set p to be $1, 0.02$ for Forms 1, 2, respectively. We present the forward invariance of the set $C_1 \cap C_2$, where $C_1 := \{\boldsymbol{x}(t) : b(\boldsymbol{x}(t)) \geq 0\}$ and $C_2 := \{\boldsymbol{x}(t) : \psi_1(\boldsymbol{x}(t)) \geq 0\}$ in Fig. 2.

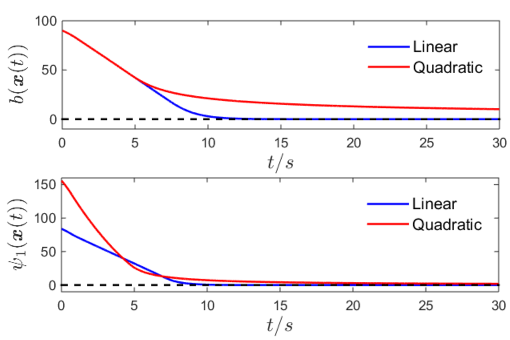


Fig. 2. $b(\boldsymbol{x}(t))$ and $\psi_1(\boldsymbol{x}(t))$ under Forms 1, 2. $b(\boldsymbol{x}(t)) \geq 0$ and $\psi_1(\boldsymbol{x}(t)) \geq 0$ imply the forward invariance of the set $C_1 \cap C_2$.

VII. CONCLUSION & FUTURE WORK

We presented an extension of control barrier functions to high order control barrier functions, which allows to deal with high relative degree systems. We also showed how we may deal with the conflict between the HOCBF constraints and the control limitations. We validated the approach by applying it to an adaptive cruise control problem with constant safety constraint. In the future, we will apply the HOCBF method to complex problems, such as differential flatness in high relative degree system and bipedal walking.

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